

J. C. G. W.

Heathcote. Marshfield

A First Course in Practical Mathematics

BY

B. A. TOMES

Headmaster, Tredworth Council School, Gloucester
Lecturer in Mathematics, Gloucester School of Science and Technology

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PREFACE

This book is compiled as the result of several years' close attention to the teaching of mathematics in Primary and Technical Schools, and is an attempt, based on practical experience, to formulate a reliable course of study which shall serve to link up the work of those two types of education. The need for some such link is emphasized in the recent reports of the Board of Education. The book is therefore designed to suit the requirements of the highest standards in the Primary School and the preliminary classes in mathematics in Evening Continuation and Technical Schools.

As the title suggests, the book is a course of study, and claims to be progressive. It is hoped that the exercises will be worked through in the order given. The examples, besides being carefully graduated, are considered just sufficient to impress the truths they illustrate. It will be found, too, that a more perfect understanding of principles, and the evolution of further mathematical truths, will attend the systematic working of the examples of the exercises in order. All branches of mathematics are considered together as far as is practicable from the matter in hand. This arrangement has been adopted in order to give special prominence to the interdependence of mathematical magnitudes and to lead to a fuller appreciation of "variables", a knowledge of which should sufficiently prepare the student to enter upon more advanced mathematical considerations.

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A FIRST COURSE IN PRACTICAL MATHEMATICS

SECTION I.—BEGINNINGS

A **line** has length without breadth. It may lie evenly between its extreme points, when it is termed a *straight line*; or unevenly, when it is termed a *curved line*.

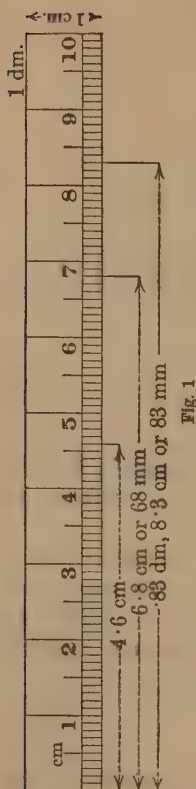
The **length** of a line is measured in terms of *yards, metres, feet, centimetres, &c.* Before proceeding to express the measure of any length, it is necessary to decide on some such *unit* of length. The mind has then some definite way of estimating and recording length in any direction.

UNITS

The **metre** is an agreed-upon unit length. Attempts were made to make it measure exactly the ten-millionth part of the distance from the Equator to the North Pole along the meridian of Paris. Recent calculations have shown a slight error in this estimation, and in consequence the “metre” is now defined as the “distance between two parallel lines crossing a bar of platinum” kept in the French archives at Paris.

The **yard** is the agreed-upon unit length commonly used in Great Britain. Each country has its own particular unit length, but the above two—the yard and the metre—are those with which we ordinarily have to deal.

The metre (abbrev. m.) is subdivided into 10 equal parts called *decimetres* (dm.). Each decimetre is subdivided into 10 *centimetres*



(cm.), and each centimetre is subdivided into 10 *millimetres* (mm.), as in fig. 1.

This mode of subdivision is also found when *inches* are divided into tenths.

Exercises 1.—Mental Exercises with Ruler

1. How many tenths of inches are there in $2\frac{3}{10}$ in.?
2. How many inches are there in 47 tenths of inches?
3. How many millimetres are there in 4 cm. 6 mm.? (Fig. 1.)
4. Convert 68 mm. to centimetres.
5. What number of millimetres are there in $9\frac{7}{10}$ cm.?
6. How many centimetres make 1 m.? Hence state the number of centimetres in 2 m. 5 dm. and in 3 m. 4 dm. 6 cm.
7. How many millimetres make 1 m.? Express as millimetres 4 m. 3 dm. 6 cm., 2 m. 4 cm. 8 mm., and 3 dm. 6 mm.

These exercises show reasons for using prefixes to indicate tenths, &c., viz. *deci-*, *centi-*, and *milli-*. “Metre” means a “measure”. *Deci-* is a Latin prefix meaning ten, and is therefore used to name “one-tenth of the measure”. *Centi-* means “one hundred” and *milli-* “one thousand”. Thus—

$$10 \text{ dm.} = 1 \text{ m.}$$

$$100 \text{ cm.} = 10 \text{ dm.} = 1 \text{ m.}$$

$$1000 \text{ mm.} = 100 \text{ cm.} = 10 \text{ dm.} = 1 \text{ m.}$$

There are measures greater than the metre, just as in British reckoning there are measures greater than the yard. Thus—

$$10 \text{ m.} = 1 \text{ decametre (Dm.);}$$

Gr. *deka-* (ten).

$$100 \text{ m.} = 10 \text{ Dm.} = 1 \text{ hectometre (Hm.);}$$

Gr. *hecto-* (hundred).

$$1000 \text{ m.} = 100 \text{ Dm.} = 10 \text{ Hm.} = 1 \text{ kilometre (Km.);}$$

Gr. *kilo-* (thousand).

Metric measurement is therefore *decimal*, i.e. the table proceeds in tens, from millimetres to kilometres, each higher unit being ten times the one it succeeds.

Exercises 2.—Mental Exercises with Metre Ruler

1. How many millimetres are there in $4\frac{6}{10}$ or 4.6 cm.? (Note how the decimal point divides units and tenths; thus, .6 means $\frac{6}{10}$ or 6 tenths.)
2. Express in centimetres, 15 mm., 625 mm., and 4385 mm. How many decimetres are there in 15 cm., 625 cm.?
3. Express in hectometres, 15 Dm., 315 Dm., 6 Km., 46 Km.
4. Express in metres, 14 Dm., 26 dm., 8 dm., 24 Dm.

[*Method question A.*—In Exercises 2, 3, and 4 above, the decimal point may be considered to be placed after each of the numbers; thus, 15 mm. = 15.0 mm., 625 cm. = 625.0 cm. (The 0 is placed after the decimal point to emphasize the fact that the point has been inserted.) How have you converted centimetres to millimetres, hectometres to decametres, and vice versa? How is this operation effected by moving the decimal point?]

5. How many metres in 6 Hm. 4 m., 6.8 Hm., 4 Hm. 3.4 Dm, 6 Hm. 2 Dm.?
6. Bring 7.36 Km. to Hm., 7 m. 6 dm. to cm., 8 dm. 4 mm. to mm.
7. Express 365 cm. as metres, 4568 mm. as decimetres, and 537 m. as hectometres.
8. Bring 6.38 Km. to Dm., 4.76 Hm. to metres, 14.3 m. to centimetres, and 16.87 dm. to millimetres.

[*Method question B.*—In Exercises 7 and 8, how did you reduce one denomination to the *next but one* higher and lower, e.g. kilometres to decametres, metres to hectometres, &c.? What movements of the decimal point effected this?]

9. Express 6 Km. 4 Hm. 3 Dm. 2 m. 5 dm. as metres, decimetres, decametres, and kilometres.
10. Express 4 m. 3 dm. 8 cm. 9 mm. as millimetres, centimetres, decimetres, and metres.
11. Express 5 Km. 4 Dm. as metres, 6 Hm. 3 m. 6 dm. as metres, 4 Dm. 3 cm. as decametres.
12. How many centimetres are there in (a) 5 m. 4 dm. 2 mm., (b) 3 m. 5 dm., (c) 4 Dm. 6 dm. 7 cm.?

From Exercises 2 you will have discovered—

1. To multiply by 10, the decimal point is moved *one place* to the *right*.

2. To divide by 10, the decimal point is moved *one place* to the *left*.

3. To multiply by 100, the decimal point is moved *two places* to the *right*.

4. To divide by 100, the decimal point is moved *two places* to the *left*, and so on.

$$\text{Thus } 7 \text{ m. } 4 \text{ dm. } 3 \text{ cm. } 9 \text{ mm.} = 7439 \text{ mm.} = 743.9 \text{ cm.} = 74.39 \text{ dm.} = 7.439 \text{ m.};$$

$$\begin{aligned} \text{and } 3 \text{ Dm. } 6 \text{ m. } 4 \text{ cm. } 2 \text{ mm.} &= 36042 \text{ mm. (0 because no} \\ &\text{dm. are stated)} \\ &= 3604.2 \text{ cm.} = 360.42 \text{ dm.,} \\ &\text{and so on.} \end{aligned}$$

WEIGHT AND CAPACITY—MONEY

As in length, so in weight and capacity, certain *units* are adopted, and these units differ in various countries.

The British standard unit of weight is the **pound** (lb.), and corresponds to the metric **gramme** or **gram**.

In Great Britain capacity is expressed in terms of **pints**, **quarts**, **pecks**, but the French use the **litre**, **kilolitre**, &c.

Similarly, the **franc**, **sou**, and **centime** correspond to our **shilling**, **penny**, and **farthing**.

All the French measures and tables are decimal, and are built up just as the metric table of length, the words "gramme" and "litre", &c., substituting "metre" in the statements. Thus—

10 milligrams	1 litre	= 10 decilitres.
(mg.) = 1 centigram (cg.)		= 100 centilitres.
10 centigrams = 1 decigram (dg.)		= 1000 millilitres.
10 decigrams = 1 gram (g.)	1 kilolitre	= 1000 litres.
10 grams = 1 decagram (Dg.)		= 100 decalitres.
10 decagrams = 1 hectogram (Hg.)		= 10 hectolitres.
10 hectograms = 1 kilogram (Kg.)		

1 franc = 10 sous or decimes = 100 centimes. The abbreviation fr. takes the place of the decimal point in stating sums of money. Thus, 6 francs 25 centimes is written 6 fr. 25.

Note.—If English money were decimalized by adding 40 far-

things to the 960 farthings making £1, the English table could, as here stated, be made to read—

$$£1 = 10s. = 100 \text{ pence} = 1000 \text{ farthings,}$$

and money reckoning could be simplified without much interference with existing financial arrangements.

Exercises 3.—Mental Exercises

1. Express 6 g. 5 dg. 3 cg. as grams.
2. 5 g. 3 dg. of water fill 5·3 millilitres. How many grams does 1 litre of water weigh?
3. Find the cost of 2·5 Kg. of tea at 3 fr. per kilogram.
4. If 6 m. of wire weigh 24 g., how many centigrams will 1 m. weigh?
5. Express 2 fr. 25 (i.e. 2 fr. 25 c.) as centimes, and find the cost of 2 m. 5 dm. of material at 2 fr. 25 per metre.
6. Add 5 g., 3 g. 2 dg., and 4 g. 7 dg. Give your result in grams.
7. From 4 m. 6 cm. take 3 dm. 8 cm. Give your result in centimetres.
8. If English money were decimalized on the above plan, express 7584 farthings as pounds, and £6, 8s. 9d. as farthings.
9. Milk sells at 0 fr. 45 per litre. What is the cost of 3 litres?
10. How many centimes are there in 6·8 fr., 4 fr. 86, 0 fr. 27?
11. I pay 432 c. for 6 books. Give the price of one book in francs, on the assumption that the books are of equal value.
12. A rod 17 m. long is divided into two parts, so that one is 5 m. 6 dm. What is the length of the other part in decametres?

The **yard** is subdivided into 3 *feet*, and each foot is further subdivided into 12 *inches*. Thus reduction cannot be accomplished by the simple movement of the decimal point. The multiples of the yard—5½ yards make 1 pole, 40 poles make 1 furlong, and 8 furlongs make 1 mile—hinder readiness of reduction, and the British boy has to learn how to convert inches to miles, and miles to yards, &c., by dividing or multiplying by the various numbers in the table. Exercises upon this table, as also upon the tables of avoirdupois and capacity, will already have been worked. There remains only to consider how these lengths, weights, &c., can be conveniently and readily expressed in decimal form.

SOME USEFUL HINTS AND EXERCISES

A.— $\frac{1}{2} = \frac{5}{10}$, i.e. if the whole is divided into 2 parts and 1 is taken, it is the same as if the whole were divided into 10 parts and 5 taken. This is readily seen in fig. 2. The shaded portion

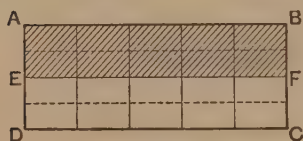


Fig. 2

ABFE is $\frac{1}{2}$ of ABCD; it is also $\frac{5}{10}$ and $\frac{10}{20}$ of the same. Therefore, $\frac{1}{2} = \frac{5}{10} = \frac{10}{20}$. This truth may be expressed thus: The numerator and denominator of any fraction may be multiplied (or divided) by the same number without effecting

any alteration in the value of the fraction.

B.— $\frac{1}{2} = \frac{5}{10}$, $\therefore \frac{1}{2} = .5$; $\frac{1}{4} = \frac{25}{100}$, $\therefore \frac{1}{4} = .25$; $\frac{1}{8} = \frac{125}{1000}$, $\therefore \frac{1}{8} = .125$. Convert the denominators of the fractions into ten or some power of ten, and the corresponding numerator gives the decimal required.

C.—(1) That $\frac{3}{8} = .375$ may be found by dividing the numerator by the denominator thus—

$$\begin{array}{r} 8 \overline{) 3.} \quad \text{Similarly } \frac{5}{12} \text{ is worked, } 12 \overline{) 5.} \\ \underline{.375.} \qquad \qquad \qquad \underline{.4166 +} \end{array}$$

and $\frac{5}{12} = .4166\dots$ or $.41\dot{6}$, where a dot is placed over the 6 to show that it repeats.

$$\begin{array}{llll} (2) \quad \frac{1}{2} = .5 & = .5. & \frac{1}{2} = .5, \therefore \frac{1}{20} = .05. \\ \frac{1}{4} = \frac{1}{2} \text{ of } \frac{1}{2} = \frac{1}{2} \text{ of } .5, & & \\ \text{i.e. } .5 \div 2 = .25. & \frac{1}{4} = .25, \therefore \frac{1}{40} = .025. \\ \frac{1}{8} = \frac{1}{2} \text{ of } \frac{1}{4} = \frac{1}{2} \text{ of } .25, & & \\ \text{i.e. } .25 \div 2 = .125. & \text{Similarly, } \frac{1}{80} = .0125, \\ \frac{1}{10} = \frac{1}{2} \text{ of } \frac{1}{8} & = .0625. & \frac{1}{160} = .00625, \\ \frac{1}{32} = \frac{1}{2} \text{ of } \frac{1}{16} & = .03125; & \frac{1}{320} = .003125; \\ \text{and so on.} & & \text{and so on.} \end{array}$$

(3) Compare—

$$\left\{ \begin{array}{l} \frac{1}{4} = .25 \text{ with } \frac{1}{40} = .025 \text{ and } \frac{1}{400} = .0025 \\ \frac{1}{8} = .125 \quad ,, \quad \frac{1}{800} = .00125 \quad ,, \quad \frac{1}{8000} = .000125 \end{array} \right\}$$

and note that the increase of the denominator to ten times its

former value reduces the value of the fraction to one-tenth its former value. This is also shown by the presence of a cipher after the decimal place.

D.—Note the following examples. The *steps* will show how to educe the *method* employed in each case.

Example 1.—Express 7 yd. 2 ft. 3 in. as yards.

$$\text{Steps—} 3 \text{ in.} = \frac{3}{12} \text{ ft.} = .25 \text{ ft.}$$

$$\therefore 2 \text{ ft. } 3 \text{ in.} = 2.25 \text{ ft.}$$

$$2.25 \text{ ft.} = \frac{2.25}{3} \text{ yd.} = .75 \text{ yd.}$$

$$\therefore 7 \text{ yd. } 2.25 \text{ ft.} = 7.75 \text{ yd.}$$

Method.

$$\begin{array}{r} 12 \overline{) 3.} \text{ in.} \\ 3 \overline{) 2.25} \text{ ft.} \\ \underline{7.75} \text{ yd.} \end{array}$$

Example 2.—£7, 16s. 8½d. as decimal of £1.

$$\text{Steps—} \frac{3}{4}d. = .75d.$$

$$\therefore 8.75d. = \frac{8.75}{12} s. = .7291s.$$

$$\begin{aligned} 16.7291s. &= £16.7291 \div 20 \\ &= £.8364 + \end{aligned}$$

$$£7.8364. \text{ (Result.)}$$

Method.

$$\begin{array}{r} 4 \overline{) 3.} \text{ farthings.} \\ 12 \overline{) 8.75} \text{ pence.} \\ 20 \overline{) 16.7291} \text{ shillings.} \\ \underline{7.8364} \end{array}$$

There is, however, a short way of converting money to the fraction of £1.

(a) Every shilling is $\frac{1}{20}$ of £1; $\therefore 2s. = £.1$, $3s. = £.15$, $4s. = £.2$; and so on.

(b) $6d. = £\frac{1}{40}$, i.e. $6d. = £.025$. $6d.$ is also 24 farthings. Thus the number of farthings is one less than the significant figures occupying the hundredths and thousandths places of the decimal. Also $9d. = £.037 = 36$ farthings, and $3\frac{3}{4}d. = £.016 = 15$ farthings; and so on. The pence and farthings can thus be expressed as a decimal by adding one to the amount expressed as farthings and setting the result down as thousandths. This rule holds between the limits of $2\frac{3}{4}d.$ and $11\frac{1}{4}d.$ From $\frac{1}{4}d.$ to $2\frac{1}{2}d.$ the farthings alone express the second and third decimal places, while $11\frac{1}{2}d.$ and $11\frac{3}{4}d.$ need two added to the farthings to give the decimal correctly.

11. What change have I from 25 fr. after paying a bill of 18 fr. 50?
12. .5 of a vertical pole is above the surface of a pond, .3 is in the water, and the rest in the mud beneath. If the pole is 12.85 m. long, state the length to be observed above the water, and the depth of the pond, to the nearest centimetre.
13. Divide 5.67 m. of wire into 3 equal parts. What is the length of each part in centimetres?
14. The 4 lengths of wire used as connections in an electric circuit are 1.6 m., 2.7 m., .85 m., and 75 cm. What is the total length of wire used?

POWERS

$10 \times 10 = 100.$ 10×10 may be written 10^2 .
Thus $10^2 = 100$, or 100 is the second power of 10.

$10 \times 10 \times 10 = 1000.$ $10 \times 10 \times 10 = 10^3$, or 10 raised to the third power. Thus $10^3 = 1000$, or 1000 is the third power of 10.

$2 \times 2 \times 2 \times 2 = 16.$ $2 \times 2 \times 2 \times 2 = 2$ raised to the fourth power, or 2^4 . Thus $2^4 = 16$, or 16 is the fourth power of 2.

The 2, 3, and 4 in 10^2 , 10^3 , and 2^4 are termed **indices** (singular **index**), and the 10, 10, and 2 are termed the **bases** on which the results or powers are calculated.

The *index* 2 shows how many times the *base* 10 has to be set down that the product may be 100. Also the *base* 2 set down 4 times, as denoted by the *index*, results, by multiplication, in 16. In the case of powers of ten the index shows how many ciphers follow 1 in order to give the required power, e.g. $10^3 = 1000$ (one and 3 ciphers).

Tables can now be formed. Thus—

$3^1 = 3.$	$5^1 = 5.$
$3^2 = 3 \times 3 = 9.$	$5^2 = 25.$
$3^3 = 27.$	$5^3 = 125.$
$3^4 = 81; \text{ and so on.}$	$5^4 = 625; \text{ and so on.}$

Example 1.—If $3^x = 81$, find the value of x .

$$81 = 3 \times 3 \times 3 \times 3 = 3^4. \quad \therefore 3^x = 3^4. \quad \therefore x = 4.$$

Example 2. What number raised to the third power results in 343?

Here $x^3 = 343$. But $7 \times 7 \times 7 = 343$. $\therefore x^3 = 7^3$.
 $\therefore x = 7$. 7 is the number.

Numbers can be expressed as **units and decimals multiplied by some power of ten**, and in this form are helpful in understanding and simplifying future work. Thus—

$$600 = 6 \times 100. \quad \therefore 600 = 6.0 \times 10^2.$$

$$563 = 5.63 \times 100. \quad \therefore 563 = 5.63 \times 10^3.$$

$$6,576,000 = 6.576 \times 10^6.$$

ROOTS

$10^2 = 100$. Then 10 is said to be the **square root of 100**. This is written—

$$\sqrt{100} = 10, \text{ or } 100^{\frac{1}{2}} = 10.$$

Similarly, $4^2 = 16$, i.e. the square root of 16 is 4, and it is written $\sqrt{16} = 4$, or $16^{\frac{1}{2}} = 4$;

also $2^5 = 32$, i.e. the fifth root of 32 is 2, and it is written $\sqrt[5]{32} = 2$, or $32^{\frac{1}{5}} = 2$;

and $3^4 = 81$, i.e. the fourth root of 81 is 3, and it is written $\sqrt[4]{81} = 3$, or $81^{\frac{1}{4}} = 3$.

To find the root, therefore, it is necessary to proceed in the opposite direction to that employed to find the powers.

With powers, the base and index are given, and the power is found from them.

With roots, the power and index are given, and it is necessary to find the base.

Example 3.—Find the 4th root of 50625, i.e. find the base which, raised to the fourth power, becomes 50625.

Here $\sqrt[4]{50625}$ or $50625^{\frac{1}{4}} = x$; otherwise $x^4 = 50625$.

Factorize 50625, setting down the prime factors.

$$50625 = 5 \times 5 \times 5 \times 5 \times 3 \times 3 \times 3 \times 3$$

$$= 5 \times 3 \times 5 \times 3 \times 5 \times 3 \times 5 \times 3$$

$$= 15 \times 15 \times 15 \times 15 = 15^4.$$

$$\therefore \sqrt[4]{50625} = 15.$$

$$\begin{array}{r} 5 \overline{) 50625} \\ 5 \overline{) 10125} \\ 5 \overline{) 2025} \\ 5 \overline{) 405} \\ 9 \overline{) 81} \\ \underline{9} \end{array}$$

Exercises 6

1. What are the values of 3^4 , 5^2 , 8^3 , 9^2 , 10^4 , 10^6 , 2^1 ?

2. Find the value of x in each of the following:—

$$10^x = 1000, \quad 3^x = 243, \quad 10^x = 10000,$$

$$7^x = 343, \quad 5^x = 15625, \quad 9^x = 6561.$$

3. What number (a) raised to the 4th power = 256?

$$(b) \quad ,, \quad ,, \quad 3rd \quad ,, \quad = 27?$$

$$(c) \quad ,, \quad ,, \quad 2nd \quad ,, \quad = 10000?$$

$$(d) \quad ,, \quad ,, \quad 3rd \quad ,, \quad = 3^3 \times 1000?$$

4. What powers of 2 are 16, 64, 256, and 512 respectively?

5. Write down ten times and 10^2 times the following numbers, first as whole numbers, then as units and decimals multiplied by a power of ten:—68, 3·6, 5·74, 46·78, 4756, 128·37.

6. Express in the form $8365 = 8 \cdot 365 \times 10^3$ the following:—683, 5436, 8365420, 5007, 50, 208, $4^4 \times 10^3$.

7. Draw a line $1 \cdot 23 \times 10^2$ mm. long. What is its length in metres?

8. What numbers are represented by 6×10^2 , $6 \cdot 8 \times 10^1$, $5 \cdot 386 \times 10^2$, $1 \cdot 2365 \times 10^2$, $1 \cdot 2365 \times 10^4$?

9. Find by factorization:

(a) The square root of 625.

(b) The square root of 676.

(c) The cube root or third root of 2744.

(d) The fourth root of 1296×10^4 .

THE RECTANGLE

Surface possesses length and breadth. Any surface enclosed by lines is termed a **figure**. Surfaces may be *plane* or *curved*. Figures are often spoken of as *plane figures*, i.e. figures on a plane; figures may also be parts of curved surfaces.

Measurement of Surface.—Just as lengths are measured in terms of unit lengths—yard, metre, &c.—so surfaces or *areas* of figures are measured in terms of unit areas.

The unit areas in general use are the *square yard* and *square metre*. These units are *derived* from the units of length. A square yard is the surface or area of a square whose sides are each 1 yd. long; a square metre is the area of a square each of whose sides is 1 m. long.

A **rectangle** is a four-sided figure with its opposite sides equal and its angles right angles. It can be divided into *units of area*,

as in fig. 3. AB is, say, 4 cm. long, and AD is 3 cm. long. The shaded square has its sides each 1 cm. long. It is therefore

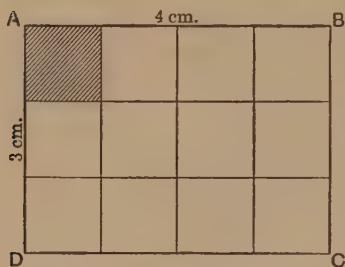


Fig. 3

1 sq. cm., and by counting these squares the area of ABCD is found to be 12 sq. cm., i.e. 12 surfaces each 1 sq. cm. in area will exactly cover the surface of the rectangle ABCD.

To discover (a) how to find the area of any rectangle; (b) how to fix the position of the decimal point in the product of a multiplication sum; and (c)

how to build up tables of area, the following exercises have been devised.

Exercises 7

1. Draw on plain paper a line 4 in. long. On it construct a square. Divide the figure into square inches on the plan of fig. 3 above. What is its area in sq. in.?
2. Repeat question 1, using lines 3 in. and 8 cm. long. By counting give their areas. How can you get these answers without counting the squares?
3. If A stands for the area of a square, and l stands for the length of its side, state in your own words the meaning of $A = l^2$.
4. Draw a rectangle 5 in. by 4 in. Divide it into square inches, and by counting state its area. How can you otherwise determine its area?
5. If A stands for the area of a rectangle, l its length, and b its breadth, state in your own words the meaning of $A = lb$ (i.e. $A = l \times b$).
6. Let 3 in. represent 1 yard. On it construct a square, and divide it into square inches. What does 1 sq. in. represent, and how many sq. ft. make 1 sq. yd.?
7. Let 3 in. represent 1 ft. Construct a square of 3 in. side and divide it into squares of $\frac{1}{4}$ in. side. What does each square represent? How many square inches make 1 sq. ft.? Show by thickening the boundary of one-fourth of the square that while 6 in. is $\cdot 5$ of 1 ft., 6 in. square is only $\cdot 25$ of 1 sq. ft. What is the result of $\cdot 5 \times \cdot 5$?
8. Draw a square decimetre and divide it into square centimetres. How many square centimetres equal 1 sq. dm., and how many square decimetres = 1 sq. m.? How many square centimetres are there in 15 sq. m. 46 sq. dm. 25 sq. cm.?

9. Draw a square of $5\frac{1}{2}$ in. side. Let A be its top left-hand corner. From A mark off the sides in inches, and by drawing lines divide the square into squares and rectangles. What fraction of a square inch is the tiny square at the opposite end of the diagonal from A? If $5\frac{1}{2}$ yd. = 1 p., how many square yards make 1 sq. p.?
10. Draw a square inch. By dividing its sides, and so the figure, into rectangles, represent (1) $\cdot 5$ sq. in., (2) $\cdot 25$ sq. in., (3) $\cdot 0625$ sq. in. In each case write down the areas, with the lengths and breadths of the rectangles, in the form $A = lb$.

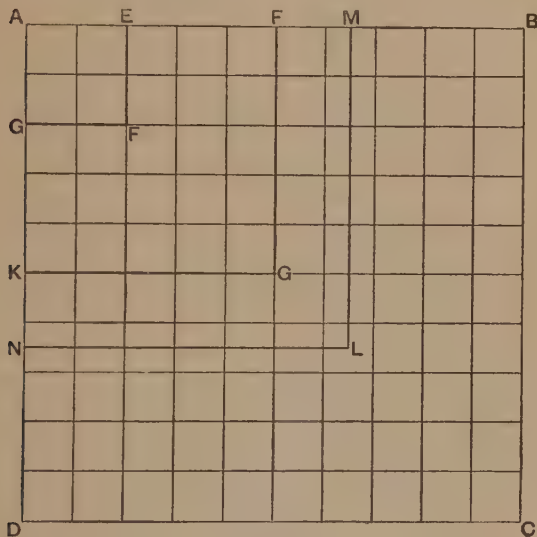


Fig. 4

11. ABCD is a square decimetre divided into square centimetres.
- What decimal fraction of AB is AE?
 - What decimal of ABCD is AEEFG?
 - How is the area of AEEFG obtained from AE?
 - When tenths are multiplied by tenths, what is the value of the product?
 - Repeat *a*, *b*, *c*, *d* with the figure AFGK.
 - What fraction of AB is AM?
 - Express the area of ANLM.
 - How can the result of *g* be obtained from the length of AM?

N.B.—All your answers are to be expressed in decimetres and square decimetres.

The method of finding the position of the decimal point in a product is also discovered from the following:—

$$68 = 6.8 \times 10. \quad \therefore 68 \div 10 = 6.8.$$

$$134 = 1.34 \times 10^2. \quad \therefore 134 \div 100 = 1.34.$$

Thus—

$$\begin{aligned} 1.34 \times 6.8 &= 68 \div 10 \times 134 \div 100, \text{ i.e. } 68 \times 134 \\ &\quad \div 10 \div 100 \\ &= 68 \times 134 \div 1000 \\ &= 8112 \div 10^3 \\ &= 8.112. \end{aligned}$$

Thus the decimal places in the multiplicand and multiplier, if added, equal the decimal places in the product.

Exercises 8

1. Find the area of a floor which is 26 ft. 3 in. long and 17 ft. 9 in. broad. Work in decimals, and express your answer as square feet.
2. Two squares have their sides respectively 15.6 cm. and 8.9 cm. Find their areas, and determine how many square centimetres one is larger than the other.
3. A garden 23.5 yd. long by 6.25 yd. broad has the garden plot in the centre 21.5 by 5.25 yd. Find the area of the paths.
4. A yard has to be tiled at the cost of 2s. 9d. per square yard. It measures 14 ft. 6 in. by 12 ft. 9 in. Find the cost.
5. A carpet 12 ft. 6 in. by 9 ft. 3 in. is placed in the centre of a floor measuring 15 ft. by 11 ft. What area needs to be stained, and determine the cost of this at 4d. per square foot?
6. What is the area of a rectangle 5.6 m. by 46.3 dm.? Give your answer in square centimetres and square metres.
7. What area of paper is needed for a wall 4.83 m. long and 38.3 dm. high?
8. Find the cost of 36.5 l. of wine at 5 fr. 25 per litre.
9. .125 Kg. of tobacco at 8 fr. 48 per kilogram.
10. If 6.45 sq. cm. make 1 sq. in., find the area of a pond in square metres if it measures 6 sq. yd. 8.6 sq. ft.

SIGNIFICANT FIGURES—APPROXIMATE MEASURES

To fully comprehend how difficult it is to obtain **exact** measures, perform the following experiments:—

Experiment 1.—Draw three lines of any lengths. Carefully measure them in millimetres by placing the scale of the ruler

right on the line. To do this the ruler must be on its edge. How many of the lines measure an *exact* number of millimetres?

Experiment 2.—Repeat 1, using tenths of an inch.

Experiment 3.—Try to express the amount by which the lines are “in excess” or “short” of the *approximate* lengths as decimals of a millimetre and of a tenth inch respectively, by dividing with the eye these small divisions of your scales into 10 equal parts.

With which scale—millimetres or tenth inches—do you expect to get the most accurate answer? Why?

You should be able to decide whether the “excess” is .25, .5, or .75 of a millimetre or $\frac{1}{10}$ in., for the eye can easily divide each small division into fourths.

Note.—(a) A line 6.4 in. and .25 of a tenth of an inch long measures 6.425 in.

(b) A line 3.2 cm. and $\frac{3}{4}$ of a millimetre long measures 3.275 cm.

(c) A line 3.7 in. all but a quarter of a tenth inch measures 3.675 in.

(d) Even if your ruler were perfect, which is impossible, you would still need a microscope to discern exact lengths to, say, four decimal places, and even then you would not be *absolutely exact*, for there would still be some excess of length which would require a fifth or sixth decimal place to express.

Our ordinary measurements are therefore only *approximate*, and are the more exact only as we use the smaller subdivisions of our scale, or have lines sufficiently long that small inaccuracies do not materially affect our results. It will be seen at once that the operations of addition, subtraction, multiplication, and division performed with approximate measurements must produce some variations in the results obtained.

The question to be decided now is: To what degree of accuracy must a quantity be measured in order that, for all practical purposes, any error made may make little or no difference to the result for which the quantity is measured.

Example 1.—An error of 1 mm. made in measuring 3 m. 8 dm. 5 cm. 4 mm., i.e. 3854 mm., will amount to 1 in 3854 (i.e. to about 1 in 4000)— $\frac{1}{4}$ per 1000, or $\frac{1}{40}$ per cent.

Example 2.—An error of 1 mm. made in measuring 5 cm. 8 mm., i.e. 58 mm., will amount to 1 in 58 (i.e. to about 2 in 100) —2 per cent.

The error in Example 1 may for ordinary purposes be considered negligible, but can on no account be overlooked in such a measure as in Example 2. Similarly, an error of $\frac{1}{2}$ in. in measuring a pole or a furlong may be without any effect on the result for which the measurement was made, but in measuring a line a few feet long the error is considerable. So, again, while a farthing or even a penny can be omitted from a large amount in pounds, it cannot be omitted from a sum of money amounting to only one or two shillings.

The following examples show the number of decimal points necessary to give practical results:—

1. 4 places of decimals enables pounds to be expressed to the nearest farthing.

- 3 places of decimals enables pounds to be expressed to the nearest penny.

£3.6542		£3.654	
20		20	
s.13.0840	£3, 13s. 1d.	s.13.080	£3, 13s. 1d.
12	(approx.).	12	(approx.).
<u>d.1.008</u>		<u>d.0.96</u>	

2. 2 decimal places enables decimals of (a) a yard, (b) a foot, (c) a metre to be expressed to the nearest inch and centimetre respectively.

3.75 yd.	
3	
2.25 ft.	3 yd. 2 ft. 3 in.
12	
<u>3.00 in.</u>	
14.65 ft.	
12	7.36 m.
<u>7.80 in.</u>	= 7 m. 3 dm. 6 cm.

3. $\left. \begin{array}{l} 3.2 \text{ ft.} = 3 \text{ ft. } 2 \text{ in.} \\ 7.4 \text{ ft.} = 7 \text{ ft. } 5 \text{ in.} \end{array} \right\} \text{ if expressed to the nearest inch.}$

Note 1.— $3.7528 = 3.75$ approximately, but $3.7568 = 3.76$ approximately.

2. Numbers can be expressed—

- (a) To four decimal places; or,
- (b) To four significant figures.

Four is mentioned because this number is sufficient for all ordinary practical purposes, but any other number could be read for four in (a) and (b) above.

3. $3.042, 46.07, 360.8,$ } are examples of *numbers expressed*
 $.04375, .008362, 1563$ } to “four significant figures”.
 $.0375, 6.3542, 7.6666$ are examples of statements to
 “four decimal places”.

Exercises 9

1. Measure the length of your paper accurately to .25 mm. and .25 tenths of an inch.
2. Express as accurately as you can (3 decimal places) the perpendicular distance between two lines on your paper.
3. Given that 1 dm. = 3.937 in., which would be more likely to be accurately expressed, (a) a line measured in millimetres, or (b) the same line measured to the nearest $\frac{1}{16}$ in.
4. Express to the nearest inch (a) 3.2 ft., (b) 4.7 ft., (c) 7.9 ft., (d) 3.4 ft.
5. Express £4.676 to the nearest penny.
6. Reduce 5.78 yd. to yards, feet, and inches, expressing your result to the nearest inch.
7. An article is priced $8\frac{1}{4}d.$ instead of $8\frac{1}{2}d.$ What does the error amount to in the sale of 1 gross?
8. I measure a line and find it 4.865 dm. long. What error per cent approximately should I have made if I had written down the length as 4.86 dm.?
9. I find the measure of a line is 3 yd. 2 ft. $3\frac{1}{2}$ in. Express this as yards, and find what decimal of a yard error would be made if the $\frac{1}{2}$ in. were omitted.
10. The measure of a line 355 mm. long is set on paper, but the 5's are indistinct. First I read it as 353 mm., and then as 335 mm. Express my error in each case as dm., and say how many times one error exceeds the other.
11. Express the following as four significant figures in the denomination mentioned: (a) £5, 18s. $8\frac{1}{4}d.$ as pounds; (b) 3 yd. 2 ft. 6 in. as yards; (c) 4 yd. 1 ft. 8 in. as feet; (d) 5 m. 3.02 dm. as metres, and (e) as centimetres; (f) £6, 13s. $6\frac{1}{2}d.$ as shillings; (g) 4 lb. 12 oz. as pounds; (h) 72 g. 3 cg. as grams; (j) 6 Dg. 5 g. 6 mg. as centigrams.

12. Express to two decimal places the approximate values of—

- | | |
|-----------------|---------------------|
| (a) 3.6582 ft. | (e) 7.252 pence. |
| (b) 3 yd. 1 ft. | (f) 8.369 m. |
| (c) £7.865. | (g) 2 ft. as yards. |
| (d) 4.326s. | (h) 8.166666 + in. |

13. What are the approximate answers of the following? (Express your answers, working mentally, giving the limits in whole numbers between which you expect the real answers to come.)

- | | |
|------------------------------|--------------------------|
| (a) 6.8342×4.0375 . | (c) $46.38 \div 2.087$. |
| (b) 5.087×40.12 . | (d) $638 \div .0027$. |

14. Where and why are the following wrong?—

- | |
|--------------------------------------|
| (a) $3.6587 = 3.65$ approx. |
| (b) $40.08 \times 2.54 = 1018.032$. |
| (c) 5 m. 3 dm. 2 mm. = 53.2 dm. |

15. A shopman in booking an order inserted 79 for 97. The articles were priced at 4s. 3 $\frac{1}{4}$ d. Find the error occasioned by the false entry.

CONTRACTED MULTIPLICATION AND DIVISION

In multiplication it is usual to start with the *units* figure, or integer of least place value, of the multiplier. Why not start with the integer of highest value? It is the one which most affects the result. Compare Methods 1 and 2.

<i>Method 1</i>		<i>Method 2</i>	
6837 $\times$ 4356		6837 $\times$ 4356	
<u>4356</u>		<u>4356</u>	
41022 = 6	times	27348 =	4000 times
34185 = 50	„	20511 =	300 „
20511 = 300	„	34185 =	50 „
27348 = 4000	„	41022 =	6 „
<u>29781972 = 4356 times.</u>		<u>29781972 = 4356 times.</u>	

In Method 1 the most important integers of the product are determined last, while in Method 2 they are determined first. Method 2 also provides a method for shortening our work and for omitting unimportant figures. The odd 972 make little difference whether inserted or omitted when 29 millions of things are considered.

The decimal point can be fixed in the first product. Its position will then cause no further worry, as the following example shows:—

$$\begin{array}{r} 4.36 \times 39.6 \\ 39.6 \end{array}$$

$$\begin{array}{r} 130.8 = 4.36 \times 30 \\ 39.24 = 4.36 \times 9 \\ 2.616 = 4.36 \times .6 \\ \hline 172.656 \end{array}$$

To get the decimal point, consider the sum as 4.36×30 .

The decimal points in the other products can be inserted beneath the first one.

$$\begin{array}{r} 4.36 \\ 30 \\ \hline 130.80 \end{array}$$

Method of Contracting Multiplication.—Consider the sum—

(1) This gives 6 integers in the answers.

(2) To give an approximately correct answer, say, to four significant figures—

$$\begin{array}{r} 73.6 \times 4.72 \\ 4.72 \end{array}$$

$$\begin{array}{r} 294.4 \\ 51.5 \\ 1.4 \\ \hline 347.3 \end{array} \begin{array}{l} 4 \text{ times.} \\ 2 \text{ } .7 \text{ times.} \\ 72 \text{ } .02 \text{ times.} \\ 92 \end{array}$$

(a) 51.52 must be made to read 51.5; and
(b) 1.472 must read 1.5.

Then the part of the sum to the right of the vertical line can be omitted and the approximate result 347.4 obtained.

This can be effected as follows:—

(a) $73.6 \times .7$ Seven times six = 42. Cross off the 6 and carry the 4, since 42 is approximately 40.
 $\begin{array}{r} 73.6 \\ .7 \\ \hline 51.5 \end{array}$ $7 \times 3 = 21, 21 + 4 = 25, \&c.$ Proceed as usual

(b) $73.6 \times .02$ The 6 is already crossed off. Take no notice of it. $2 \times 3 = 6$. Cross off the 3. 6 approximates to 10. Carry 1. $2 \times 7 + 1 = 15$.
 $\begin{array}{r} 73.6 \\ .02 \\ \hline 1.5 \end{array}$ Ans. 1.5.

Combined in one sum—

To obtain:—

$$\begin{array}{r} 73.6 \times 4.72 \\ 4.72 \\ 294.4 \dots (a) \\ 51.5 \dots (b) \\ 1.5 \dots (c) \\ \hline 347.4 \end{array}$$

(a) $\times 4$. Fix the decimal point by regarding the sum as 73.6×4 .

(b) Cross off 6; but since $7 \times 6 = 42$ (40 approx.), carry 4, and proceed $7 \times 3 + 4 = 25$; place the 5 under the 4, and so on.

(c) Cross off 3; but since $2 \times 3 = 6$ (10 approx.), carry 1.

Again, 436.7×28.72 expressed to the nearest whole number is worked thus—

To obtain:—

436.7×28.72	$400 \times 20 = 8000.$
<u>28.72</u>	(a) Cross off 7; but $8 \times 7 = 56$ (60 approx.),
<u>8734... (a)</u>	$\therefore$ carry 6.
<u>3494... (b)</u>	(c) Cross off 6; but $7 \times 6 = 42$ (40 approx.),
<u>305... (c)</u>	$\therefore$ carry 4.
<u>9... (d)</u>	(d) Cross off 3; but $2 \times 3 = 6$ (10 approx.),
<u><u>12542</u></u>	$\therefore$ carry 1.

If 0 should occur in the multiplier after you have commenced crossing off figures with succeeding multiplications, cross off a figure for it as for the other multipliers, and proceed with the next multiplier.

It is important to note that when crossing off in the multiplicand commences, it is necessary to cross off one for every multiplier used.

Method of Contracting Division.—Utilize contracted multiplication instead of bringing down ciphers after the last significant figure has been used.

$28.72) 436.7 (15.2$	The position of the decimal point is found thus—
<u>2872</u>	28 is approximately 3 tens;
<u>1495</u>	436 " " 4 hundreds.
<u>1436</u> cross off 2.	
<u>59</u>	The result of $400 \div 30 =$ tens, i.e.
<u>57</u> cross off 7.	4 hundreds divided by 3 tens give an
<u>2</u>	answer in which the first significant figure
<u>2</u>	is <i>tens</i> .

Exercises 10

Work accurately to 4 significant figures—

- | | |
|--------------------------|--------------------------|
| 1. $5.462 \times 1.547.$ | 3. $43.75 \times 28.62.$ |
| 2. $576 \times 4.328.$ | 30.97 $\times$ 17.63. |
| 7.341 $\times$ 1.836. | 70.83 $\times$ 10.25. |
| 2. 58.43 $\times$ 1.837. | 4. 467.8 $\times$ 1.358. |
| 27.65 $\times$ 2.385. | 3.684 $\times$ 236.5. |
| 31.84 $\times$ 3.65 | 5.892 $\times$ 29.43. |

5. $.1852 \times 6.327.$

$.9358 \times 1.642.$

7. $.326 \times .2365.$

6. $.1427 \times .6385.$

$.4359 \times .788.$

$.0832 \times .6097.$

7. $5.437 \times .04568.$

$.73575 \times .54061.$

$.84001 \times 3.0874.$

8. $6.07 \times .0035.$

$4.008 \times 2.005.$

$748.6 \times .003026.$

9. $120.95 \div 4.482.$

$34.621 \div 5.713.$

$142.87 \div 4.375.$

10. $27.384 \div .0317.$

$168.2 \div .843.$

$470.8 \div .2653.$

11. $176.324 \div .00837.$

$37.35 \div 536.$

$.08573 \div 14.62.$

12. $6.358 \times 72.9 \div 14.6.$

$.003572 \div .0843.$

$14.59 \times .0083 \div 5.68.$

UNITS OF VOLUME, WEIGHT, AND CAPACITY

Volume has length, breadth, and thickness. All objects, whatever they are, have volume. The amount of space within the surfaces of any object such as a cube or a sphere is termed its **volume**. Volume is measured in terms of *unit volumes*, those commonly accepted being the **cubic yard**, **cubic foot**, &c., **cubic metre**, **cubic centimetre**, &c. These units are like the units of area derived from the units of length. A **cubic foot** may be defined as the space occupied by a cube of 1 ft. edge.

In fig. 5, AB is taken to represent 1 yd. divided into 3 ft.

Then ABCD represents 1 sq. yd., and each of the nine equal squares represents 1 sq. ft.

Let these areas have thickness given them. If the thickness is 1 ft., nine *cubes* result each of 1 ft. edge.

Thus the shaded cube A represents 1 c. ft.

But the figure BCDEFG represents a cubic yard.

It is easy to see that three layers of 9 c. ft. will be required to make the cubic yard.

It follows then that 3 lin. ft. make 1 lin. yd.,

3^2 or 9 sq. ft. „ 1 sq. yd.,

and 3^3 or 27 c. ft. „ 1 c. yd.

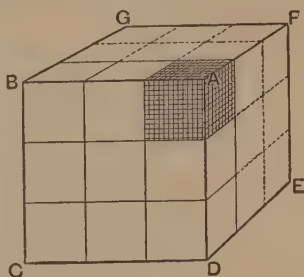


Fig. 5

Exercises IIA.—Mental Exercises

1. You have a box in the shape of a cube, each internal edge measuring one foot, also a number of blocks a cubic inch in shape and size. How many blocks placed in a row will make one edge of the box? and how many fixed side by side will cover the bottom of the box? How many layers of the cubes will be needed to fill the box? and what is the total number required?
2. Make a cubic decimetre in cardboard and a cubic centimetre in soap. How many c.c. (cubic centimetres) of soap will be required to fill a space equal to the cardboard cube?
3. How many cubic millimetres make 1 c.c.? and how many cubic centimetres make 1 c.m.?
4. How many cubic inches are there in 1.5 c. ft. and in .5 c. ft.?
5. A graduated measuring glass reads cubic centimetres. It is partly filled with water, and the surface of the water reaches the 12.5 c.c. mark. A marble is dropped into the glass, and the water rises to the 16.75 c.c. mark. What is the probable volume of the marble? How high would the water stand if two more marbles of equal size were added to the one already in the glass?
6. A *litre* is a cubic decimetre. What volume of wine occupies a Hecto-litre barrel?
7. How many cubic centimetres make 1 litre?
8. 1 c. ft. of water weighs 1000 oz. What weight of water will fill a tank 2 ft. long, 1 ft. wide, and 3 ft. deep? If the tank's depth were increased 6 in., what additional weight of water would it hold?
9. A gramme weight is the weight of 1 c.c. of water. What will 1 c.dm. of water weigh and 1 c.m. of water weigh?
10. A measuring glass is graduated to read cubic centimetres. 25.6 c.c. of water are measured out, and then 25.6 c.c. of an oil only three-quarters as heavy as water. Give the weights of liquid measured in each case.
11. Iron is 7.2 times heavier than water. Given that 1 c.c. of water weighs 1 g., find the weight of 15 c.c. of iron.
12. Iron is 7.2 times heavier than water, bulk for bulk, and 1 c. ft. of water weighs 1000 oz. Give in ounces the weight of 3.5 c. ft. of iron. What length of iron must be cut off from an iron bar of .5 sq. ft. in cross-section that I may obtain a piece weighing 14,400 oz.?

Exercises IIB

1. 1 c. in. of cast iron weighs .26 lb. Find the weight of an iron bar 7 ft. long, .3 ft. broad, and .25 ft. thick.
2. The weight of a brass rod is 6.73 lb., and 1 c. in. of brass weighs .298 lb. What is the volume of the rod?

3. 1 c.c. of water weighs 1 g. Find the volume of a casting of iron in cubic metres which weighs 437.5 Kg. (Iron is 7.2 times heavier than water.)
4. What is the volume of a beam of wood whose cross-section measures 58.75 sq. in., and whose length is 7 ft. 8 in.?
5. A marble is dropped into water in a measuring glass graduated to read cubic centimetres, and the water rises from the 15.8 c.c. mark to the 18.4 c.c. mark. If the substance of which the marble is composed is 2.6 times heavier than water, bulk for bulk, and 1 c.c. of water weighs 1 g., find the weight in grammes of the marble.

Exercises 12.—General Exercises on the Work of the Section

1. Measure the length and breadth of your paper as accurately as you can in inches and centimetres, and determine its area in square inches and square centimetres.
2. Express to the nearest third decimal place the area of a square of 3 in. side in square centimetres.
3. Find the area of a rectangle 3.65 in. long by 4.75 in. broad.
4. Find the value, to four significant figures, of 9.325×2.056 and $9.325 \div 2.056$.
5. How many lengths of .075 in. can be marked on a length of 20.875 in.?
6. Find the value of 6.385×10^2 multiplied by 4.254×10 .
7. Express correctly, to four significant figures, $6.3872 \times 10^3 \div 1.4763 \times 10^2$.
8. A plot of ground is 97.63 in. long and 1723 cm. broad. Find its area.
9. Sound travels at the rate of 1.125×10^3 ft. per sec., and the report of a gun was heard 2.75 sec. after the flash was seen. If the time taken for the light of the flash to travel to the observer be neglected, how far is the gun away?
10. Find x in: $x^6 = 64$; $4^x = 64$; $x = 3.16 \times 4.375$.
11. Light travels at the rate of 1.86×10^8 m. per second. How long will light take to travel from the sun to the earth, a distance of 9.3×10^7 m.?
12. State by inspection how you know the following to be incorrect:—

(a) $3.14 \times 4.38 = 137.53$.

(b) $63.8 \div 17.2 = .37$.

(c) £3, 6s. 9½d. = £3.336.

(d) $4.387 = 4.3$ approx.

(e) $\frac{1}{16} = .025$.

SUMMARY

A **line** has length only, and its measure is taken by means of *unit lengths*, e.g. metre, centimetre, yard, inch.

A **surface** has length and breadth only. Its *area* is measured in terms of *unit areas*. These are *derived* from the units of length, e.g. square metre, square yard.

A **volume** comprehends length, breadth, and thickness, and *unit volumes* are derived from unit lengths. Volumes of substances have *weight* measured in grammes, pounds, &c., and *capacity* (i.e. they fill space), which is measured in litres, pints, &c.

The *metric system* of weights and measures is based on the decimal. The standard units are: Length, metre; capacity, litre; weight, gramme. The prefixes *deci-*, *centi-*, *milli-* indicate .1, .01, and .001 of these units, and the prefixes *Deca-*, *Hecto-*, *Kilo-* mark 10, 100, 1000 times the units.

Place value.—The decimal point is a means of marking where the fractional part begins. All integers to the left of it are whole numbers, and all to the right are fractions. Reading to the left, if an integer is placed in the first place, it is *units*; if in the second, it is *tens* (i.e. ten times its units value); if in the third place, *hundreds*; and so on, each place farther to the left making it ten times the value it possessed in its preceding position. Reading to the right, an integer in the first place is *tenths*; in the second, *hundredths*; in the third, *thousandths*; and so on, becoming ten times less in value in each succeeding position.

The **power** of a number is the result of setting down that number a given number of times and finding the product.

The **root** of a number is that number which, when set down a certain number of times and multiplied together, gives that number.

E.g. the second *power* of 3 or $3^2 = 3 \times 3 = 9$. The cube *root* of 8 or $\sqrt[3]{8} = 2$, for $2 \times 2 \times 2 = 8$.

Section I gives methods for expressing numbers approximately, as decimals, and for contracting multiplication and division.

Points to be remembered.—

1. Area of a rectangle: $A = lb$, where A = area, l = length, b = breadth.

2. 1 c.c. of water weighs 1 g.; 1 c. ft. of water weighs 1000 oz.

3. 1 c.dm. occupies 1 litre of space.

4. To multiply by a *power* of ten, move the decimal point that number of times to the right.

SECTION II.—THE EQUATION $ab = cd$

THE MEANING OF $ab = cd$

(a) You will agree that

$$6 \times 3 = 9 \times 2 \text{ and } 4 \times 3 = 2 \times 6.$$

The sign $=$ is placed between two equal products, and such statements are termed **equations**. We need a general method of stating that two products are equal to one another, and for this purpose letters are used, a letter representing each one of the factors in the equal products. Thus $a \times b = c \times d$ represents $6 \times 3 = 9 \times 2$, $4 \times 3 = 2 \times 6$, or any other equality of products.

(b) The multiplication sign, which is necessary in arithmetic, need not be always expressed when letters are used. ab means $a \times b$, and $6ab$ means $6 \times a \times b$. (37 does not mean 3×7 . 37 means $30 + 7$ or $3 \times 10 + 7$, because numbers are placed according to their place value.) There is no place value given to letters. Thus $37 \times x \times y \times z$ can be written $37xyz$, and $a \times b = c \times d$ may be written $ab = cd$.

The statement $ab = cd$ is an *equation*, and is a *general* way of setting down the idea that two products are equal to one another.

HOW $ab = cd$ CAN BE MANIPULATED

$4 \times 6 = 3 \times 8$. Now divide each side by 4, since the *quarters* of equal things must be equal. Then we get—

$$\frac{4 \times 6}{4} = \frac{3 \times 8}{4}, \text{ i.e. } 6 = \frac{3 \times 8}{4}.$$

You will observe that by this manipulation the figure 4 has gone from the *left-hand* side of the equation, where it is a *multiplier*, and appears on the *right-hand* side as a *divisor*. Similarly—

$$4 \times 6 = 3 \times 8. \quad \therefore \frac{4 \times 6}{6} = \frac{3 \times 8}{6}, \text{ i.e. } 4 = \frac{3 \times 8}{6},$$

and 6 is transferred. Put in *general* terms, this is expressed thus: dividing each side of the general equation $ab = cd$ by a , we get—

$$\frac{ab}{a} = \frac{cd}{a}, \text{ i.e. } b = \frac{cd}{a},$$

and a is transferred. Similarly, dividing each side by b , c , and d in turn, we get—

$$a = \frac{cd}{b}, \quad \frac{ab}{c} = d, \text{ and } \frac{ab}{d} = c.$$

In every case it is noted that when the letter is transferred across the signs of equality it ceases to be a *multiplier* and becomes a *divisor*.

Example 1.—If $a = 6$, $b = 3$, and $c = 9$, find d in the statement $ab = cd$.

Method 1.— $ab = cd$. Substituting for a , b , and c .

$$6 \times 3 = 9 \times d, \text{ i.e. } 9d = 18. \quad \therefore d = \frac{18}{9} = \underline{\underline{2}}.$$

Method 2.— $ab = cd$. Dividing both sides by c .

$$\frac{ab}{c} = d, \text{ or } d = \frac{ab}{c}. \text{ Substituting for } a, b, \text{ and } c.$$

$$d = \frac{ab}{c} = \frac{6 \times 3}{9} = 2. \quad \therefore d = \underline{\underline{2}}.$$

Example 2.—If $x = 7$, $y = 9$, and $a = 3$, find b in the expression $ab = xy$.

Method.— $ab = xy$.

$$\therefore b = \frac{xy}{a} = \frac{7 \times 9}{3} = 21. \quad \underline{\underline{b = 21.}}$$

Exercises 13

1. If $a = 3$, $b = 2$, and $c = 4$, find the values of: $7a$, $12bc$, $4ac$, $7abc$, $6a^2$ ($a^2 = a \times a$; Sec. I), $4a^2b$, abc .
2. Find x in the equations: $14x = 7 \times 36$, $12x = 1728$, $14 \cdot 3x = 27 \cdot 68$.
3. Given that $ab = cd$, find the value of a when—

- (a) $b = 36$, $c = 45$, and $d = 20$.
- (b) $b = 36$, $c = 4 \cdot 5$, and $d = 9 \cdot 7$.
- (c) $b = 4 \cdot 7$, $c = 18 \cdot 6$, and $d = \cdot 095$.
- (d) $b = 16$, $c = 1 \cdot 72$, and $d = 18 \cdot 38$.

4. What is the value of z when $y = 28 \cdot 7$ and $x = 14 \cdot 35$, and the relationship between x , y , and z is expressed by the equation $x^2 = 3yz$?
5. Show by examples that if in $\frac{a}{b} = \frac{c}{d}$ each side is multiplied by bd , $ad = cb$. What becomes of the divisors b and d if they are transferred across the signs of equality?
6. Given that $\frac{a}{b} = \frac{c}{d}$, find the value of a when $b = 7$, $c = 17$, and $d = 34$, and the value of d when $a = 16 \cdot 5$, $b = 14 \cdot 7$, and $c = 10$.
7. What is the value of x in the following?—

$$(a) \frac{x}{3 \cdot 2} = \frac{6 \cdot 7}{12}$$

$$(b) \frac{x}{4 \cdot 382} = \frac{2 \cdot 637}{4 \cdot 759}$$

$$(c) \frac{3 \cdot 62}{x} = \frac{10 \cdot 86}{144 \cdot 72}$$

$$(d) \frac{\cdot 086}{1 \cdot 385} = \frac{x}{4 \cdot 155}$$

$$(e) \frac{67}{14 \cdot 28} = \frac{6 \cdot 7}{x}$$

$$(f) \frac{123 \cdot 9}{14 \cdot 08} = \frac{\cdot 0936}{x}$$

$$(g) \frac{1 \cdot 47 \times 10^2}{3 \cdot 58 \times 10^3} = \frac{x}{4 \cdot 69 \times 10^2}$$

$$(h) \frac{x}{\cdot 00002117} = \frac{100}{\cdot 002}$$

(Use contracted methods wherever possible.)

Roots.—*Example 3.*—Find $\sqrt[3]{abc^2}$ if $a = 3$, $b = 2$, $c = 6$.

$$\text{Method.}—\sqrt[3]{abc^2} = \sqrt[3]{3 \times 2 \times 6 \times 6} = \sqrt[3]{6^3} = \underline{\underline{6}}.$$

Example 4.— $x^2 = abc$. Find x when $a = 1$, $b = 2$, and $c = 16$.

$$x^2 = 1 \times 2 \times 16 = 32, \text{ i.e. } x = \underline{\underline{\sqrt{32}}}.$$

At this stage it is impossible to explain the process for finding the square root of any number, but it will be convenient to know the method. The following will make the process clear:—

Find the square root of 63,542.568.

$$\begin{array}{r}
 \text{Method.} \quad \left| \begin{array}{l} 6,35,42, \cdot 56,8 \quad (\quad 2 \quad 5 \quad 2 \cdot 0 \quad 7 \\ 4 \\ \hline 2 \times 2 \quad 4 \quad 5 \quad \hline 235 \\ 225 \\ \hline 25 \times 2 \quad 5 \quad 0 \quad 2 \quad \hline 1042 \\ 1004 \\ \hline 252 \times 2 \quad 5 \quad 0 \quad 4 \quad 0 \quad 7 \quad \hline 38 \cdot 5680 \\ 35 \quad 2849 \\ \hline 3 \quad 284100 \end{array} \right.
 \end{array}$$

Steps.—(1) Divide the numbers into periods of 2, beginning at the decimal point.

(2) Find the nearest square root to 6, set it in the answer, square it, and subtract from 6. $2^2 = 4$; $6 - 4 = 2$.

(3) Bring down the next period.

(4) Multiply the number in the answer by 2 and set the 4 which results on the left. $2 \times 2 = 4$.

(5) Obtain trial figure by dividing this number into the number obtained under (3) without the units figure, $23 \div 4 = 5$. Set this figure by the number and in the answer.

(6) Multiply 45 by 5, subtract, bring down next period, multiply answer by 2, and proceed as in 3, 4, and 5 again.

N.B.—1. When the number forming the divisor is greater than the number into which it is being divided, 0 is put in the answer and also on the right of the divisor. Then the next period is brought down, and the sum proceeded with in the usual way.

2. A period consists of two figures. If at the end of the decimal figures there is only one figure in the period, a cipher is put after it to make it a period of two.

Exercises 14

- Find the value of $\sqrt{3x^2y}$ if $x = 2$ and $y = 3$.
- If $a = 2$, $b = 3$, $c = 6$, $d = 5$, and $e = 0$, find the values of:
 - $7ab^2c$, (b) $4acd^2$, (c) $\sqrt{abc}$, (d) $3a^2e$, (e) a^3b^2cd , (f) $\sqrt[3]{3a^3b^2}$, (g) $\sqrt[4]{7e^3}$.
- Find the square roots of: 36, $36a^2$, 1476, 315.87, $144a^2b^4$.
- What is the square root of 31875.423.

5. Find x in the following:—

$$(a) x^3 = 683.8225.$$

$$(b) \frac{17.3}{x} = \frac{x}{14.62}.$$

$$(c) \frac{2.381 \times 10^2}{x} = \frac{x}{4.763 \times 10}$$

6. Given that $x^2y = 3z$, find the value of x when $y = .025$ and $z = 27.38$.

SOME USES OF THE EQUATION $ab = cd$

1. **The Area of a Rectangle.**—The area of a rectangle has already been found to be the product of its length and breadth (Section I), and can be expressed

$$A = lb.$$

As the *length* and *breadth* of a rectangle *vary*, so the *area* varies, a small area being the result of having a rectangle of short length and breadth, and a large area resulting from a rectangle having great length and breadth.

Note.—This **dependence** of the *area* of a rectangle on its *length* and *breadth* is worth noting, for the sake of understanding future work. Suppose a number of rectangles are drawn all 3 in. broad but of different lengths. If the first were 4 in. long, its area would be 12 sq. in., since $A = lb = 4 \times 3 = 12$ sq. in.; if the second were 6 in. long, its area would be 18 sq. in.; if the third were 10 in. long, its area would be 10×3 sq. in., or 30 sq. in., and so on. In every case the area would *vary* with the length of the rectangle, increasing or decreasing as its length was increased or decreased. This idea of variation is shown by the sign $\propto$, which means *varies as*, just as the sign $=$ means *equal to*.

A $\propto l$ means “the area of the rectangle varies as its length”. The area also *varies as* the breadth, or $A \propto b$.

The area, length, and breadth of a rectangle may be termed the “variables”, and it is easy to see that the result obtained in every case depends on what values are given to these “varying quantities”.

If $A = lb$, then—

$$\frac{A}{l} = b, \text{ or } b = \frac{A}{l}; \text{ also } \frac{A}{b} = l, \text{ or } l = \frac{A}{b}$$

by Exercise 9, for $A = lb$ is like the equation $ab = cd$. Thus the *length* of a rectangle is found by dividing the *area* by its *breadth*, and the *breadth* is found by dividing its *area* by its *length*.

An equation showing in general terms the relationship among variables is termed a **formula**. In using a formula it is always an advantage to state it with the *unknown quantity* only on the left-hand side of the “equals” sign.

Example 1.—Find the breadth of a rectangular garden whose area is 204.9 sq. m., and whose length is 16.73 m.

$A = lb$. In the sum, b is the *unknown quantity* or quantity to be found. Now $b =$

16.73 204.9	b	$\frac{A}{l}$
--------------------	-----	---------------

gives the formula with the *unknown* b on the left, and is found by transferring l to the other side of the equation. Then—

$$b = \frac{A}{l} = \frac{204.9 \text{ sq. m.}}{16.73 \text{ m.}} = \underline{\underline{12.25 \text{ m.}}}$$

Example 2.—What length of paper 2 ft. wide is required to cover a wall 24 ft. long by 9 ft. high?

The area of the paper (P) = the area of the wall (A).

Also $A = lb$ where l and b are the length and breadth of the wall, and $P = cd$ where c and d are the length and breadth of the paper. Thus if—

$$\begin{aligned} P &= A, \\ cd &= lb, \\ c &= \frac{lb}{d} = \frac{24 \times 9}{2} = 108 \text{ ft.} \end{aligned}$$

Exercises 15A

1. The length of a rectangle is 7.5 ft., and its breadth is 2.75 ft. Find its area.
2. Find the area of a field 116.73 m. long by 87.35 m. broad on the assumption that it is rectangular in shape.
3. A room 17.25 ft. long has an area of 168.3 sq. ft. Find its breadth. How many boards 6 in. wide will be needed side by side to floor it?

4. Find the area of a square metal plate of 36.5 cm. side.
5. A yard measures 16.5 ft. by 14.3 ft. Find the cost of paving it at 3s. 3d. per sq. yd.
6. What is the length of the side of a square plot of land whose area is 1437.5 sq. m.?
7. The length of a deal board (rectangular in shape) is 8 m., and its surface area is 54.6 sq. cm. Find its breadth.
8. Draw a rectangle 5 in. by 4 in. and divide it into 4 strips each 1 in. wide. Express the area of the rectangle in square centimetres, the width of each strip in centimetres, and the total length of paper strips required to cover the rectangle in inches and centimetres.

Exercises 15B

1. Find the cost of covering a court 15 yd. 2 ft. long by 11 yd. 1 ft. 3 in. broad with gravel at 1s. 6d. per sq. yd.
2. Find the cost of paving a road 140 yd. long by 2 yd. wide at 2s. 6d. per sq. yd.
3. How many blue bricks 9 in. by $4\frac{1}{2}$ in. will be required to cover a yard 15 ft. by 10 ft.?
4. A yard 14 ft. 3 in. by 8 ft. 9 in. is to be covered with tiles, each measuring 9 in. by $4\frac{1}{2}$ in. How many are needed?
5. How many postage stamps 2 cm. by 2.5 cm. will be wanted to cover an area 5 dm. by 2 dm.?
6. A box is 23.6 in. long, 13.5 in. broad, and 10 in. deep. If a tape is passed round the 4 sides of the box, what does it measure? The bottom is knocked out, and the 4 sides unnailed and placed side by side. Give the dimensions and area of the rectangle so formed.
7. Find the area of the four walls of a room 16 ft. long, 14 ft. broad, and 9 ft. high.
8. Find the length of wallpaper 2 ft. 3 in. wide needed to cover the walls of the room 28 ft. 6 in. long, 17 ft. 6 in. broad, and 12 ft. high.
9. Find the cost of covering a room 20 ft. 3 in. long by 17 ft. 6 in. broad with carpet 27 in. wide at 5s. 6d. per running yard.

2. Other Equations (Formulæ) in the form $ab = cd$ derived from First Principles.

(a) PREPARATION.—An angle is the inclination of two lines to one another which meet in a point. ABC is an angle, and is written briefly " $\angle ABC$ " or even " $\angle B$ " ($\angle$ means "the angle"). The angle formed by the meeting of *perpendicular* lines is a *right angle*, a larger angle than a right angle is termed *obtuse*, and a smaller angle *acute*.

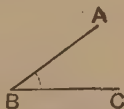


Fig. 6

Angles are variously measured, as will be seen in Section IV. The most common way is to measure it in "degrees". 90 degrees make 1 right angle.

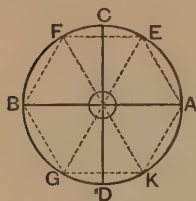


Fig. 7

The circle (fig. 7) is divided into four *quadrants* at the centre O, and the four angles made at O by the intersection of AB and CD are right angles. If the *radius* OA be struck off around the *circumference* and the points AEFBGK so obtained be joined up as in fig. 7, a *hexagon* results which is divided into six *equilateral triangles*. Six equal angles are also formed at O, each of which is therefore $\frac{1}{6}$ of four right angles, i.e. 60° . This gives the clue to the construction of

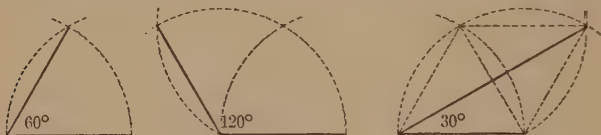


Fig. 8

angles by geometry as in fig. 8. The construction of these angles is self-evident.

A **protractor** is essentially a semicircle whose arc is divided into 180 parts. Each point on the semicircumference is projected towards the centre.

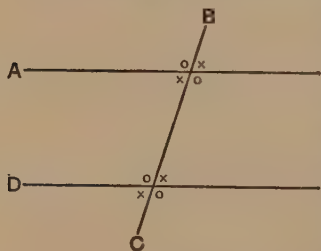


Fig. 9

When any line BC crosses two *parallel* lines A and D, eight angles are formed. These can be divided into two sets of four angles, of which the angles in either set are equal to one another. The equal angles are marked with like signs in fig. 9.

Exercises 16.—Geometry

1. Construct an equilateral triangle of 4-in. side.
2. On each side of a line 2 in. long construct equilateral triangles. Your result is a rhombus. Draw its other diagonal. How do the diagonals behave to one another?

3. Bisect the straight line AB, which is any convenient length, using the knowledge you have gained in question 2 to do it.
4. Construct an angle of 60° .
5. ABC is an equilateral triangle of 3-in. side. On AB another equilateral triangle ABD is constructed. Join BD. What does BD do to the angle ACB? Evolve a method for constructing an angle of 30° .
6. Construct angles of 30° , 90° ($60^\circ + 30^\circ$), 120° ($60^\circ + 60^\circ$), and 45° ($\frac{1}{2}$ of 90°).
7. Construct an equilateral triangle whose altitude (perpendicular height) is 3.5 in. *Note.*—This can be done by constructing angles of 30° and 90° at the ends. Draw a rough sketch first, and the method will become quite clear.

(b) EXPERIMENTS TO DISCOVER THE PRINCIPLES ON WHICH YOUR FORMULÆ ARE BASED.

Exercises 17

1. Construct carefully a parallelogram ABCD, having AB 4 in. long, AD 3 in. long, and the angle ADC equal to 60° . From A drop a perpendicular AE to the opposite side DC and carefully measure it. Cut out the figure, and then cut off the triangle ADE and place it with its side AD touching the side BC of the trapezium remaining.
 - (a) Into what is your parallelogram converted?
 - (b) What is the area of the figure you have formed?
 - (c) How then can you find the area of the original parallelogram?
2. Repeat question 1 with any parallelogram, and state (a) the figure into which every parallelogram can be converted by slicing off a *right-angled triangle* and fitting it to the other end of the figure, and (b) what relationship must exist between the areas of parallelograms constructed on equal bases and of equal perpendicular height.
3. ABCD is a parallelogram, AB is 3 in. long, AD is 2 in., and $\angle ADC$ is 60° . Cut out the figure, and cut it into two parts along its *diagonal* AC. Superimpose the two triangles, and state your observations. Would you have had a similar result if you had cut the figure along the diagonal BD.
4. Construct a triangle whose sides BC, BA, and AC are 4 in., 3 in., and 2.5 in. respectively. From A and C strike off arcs of radius 4 in. and 3 in. respectively, which intersect in D. Join DA and DC. State (a) the name of the figure ABCD, (b) the relationship its area bears to the area of the triangle.
5. Draw a square of 3 in. side. By drawing its diagonals you form 4 isosceles triangles. How many times is the area of the square larger than the area of each triangle? Find the area of each triangle.

6. By means of a piece of thread or a tape measure obtain the circumference of a jar or tin (cylinder). Find also by careful measurement the diameter of the base. Divide the circumference by the diameter. Repeat your work with cylindrical vessels of different sizes. (a) Do your answers vary much? (b) What is your approximate result? (c) How would you propose to find the circumference of a circle, given its diameter?
7. Draw a circle on cardboard. Cut it out. Place a point A on its circumference. By rolling it on its edge on a ruler find the length of the circumference. Halve the circle and measure the diameter. Divide the length of the circumference by the length of the diameter. What is your result? Repeat with a larger circle. Which result would you expect to be more reliable, and why?
8. On squared paper draw concentric circles having their radii 10, 20, and 30 squares respectively. On these radii construct squares. By counting the squares determine the areas of the circle and

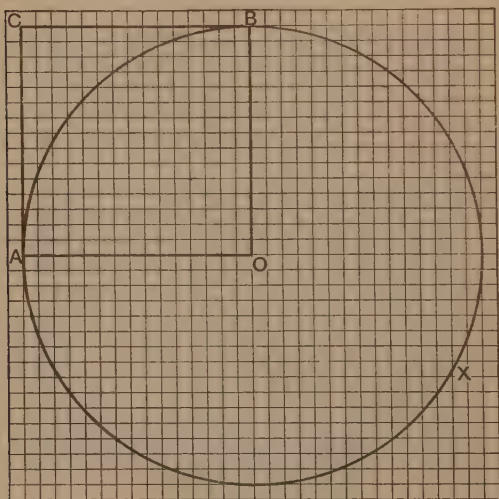


Fig. 10

Area of square = 225 squares.

Area of circle = $(176 \times 4) = 704$ squares.

∴ Relationship between square on radius and the area of the circle

$$= \frac{704}{225} = 3.13 \text{ approx.} \dots\dots\dots (\pi)$$

Note.—In counting the squares in the circle quadrant—

- | | |
|--|---|
| (1) Call $\frac{1}{2}$ square a half square | } where the circumference crosses
the squares. |
| (2) Call more than $\frac{1}{2}$ square 1 square | |
| (3) Neglect less than $\frac{1}{2}$ square | |

the square. Divide the former by the latter. In which case is your answer most accurate? Why are they approximately the same result?

N.B.—To discover a method for counting the squares in a circle, see fig. 10.

9. Take a line 40 squares long. On it construct a square, and in the square inscribe a circle. Determine the areas of circle and square by counting the squares, and again divide the area of circle by that of square. What is your result? What fraction is it of the results you obtained in question 8? How many times larger is this square on the circle's diameter than the square on the radius used in the last question?

The above questions have led to an understanding of the following truths:—

1. A parallelogram is the same area as a rectangle, on the same base, and of the same perpendicular height.
2. Parallelograms of equal bases and altitudes are equal in area.
3. Every triangle is half the area of a parallelogram, and is therefore half the area of a rectangle whose length is equal to the base of the triangle and is of equal altitude.
4. The circumference of a circle is always 3.1416 times the diameter.
5. The area of a circle is always 3.1416 times the area of the square on its radius, and .7854 times (or $\frac{1}{4}$ of 3.1416 times) the area of the square on its diameter.

(c) FORMULÆ OBTAINED.—These truths enable us to discover formulæ which can be used like $ab = cd$.

(i) *Areas of Rectangle and Parallelogram.*— $A = ab$ where A = area, a = length, and b = breadth.

The area of the parallelogram $EFCB$ = area of rectangle $ABCD$. (Question 1, Ex. 13.)

(ii) *Area of Triangle.*— $A = \frac{1}{2}ab$ where A = area, a = base, and b = height or altitude.

The area of the triangle $ABC = \frac{1}{2}$ the area of the parallelogram $ABCD$.
 $\therefore$ its area = $\frac{1}{2} ab$. (Question 3, Ex. 13.)



Fig. 11



Fig. 12

Note.—The area of a triangle varies as the lengths of its base and height, or $A \propto a$ and $A \propto b$. The area, base, and height are the *variables*, and the dimensions of one depend on those of the other two. The $\frac{1}{2}$, permitting the equation to be stated, is *constant*, and appears in determining the area of every triangle. Thus if $A \propto ab$ and the constant quantity $\frac{1}{2}$ permits the formula to be written $A = \frac{1}{2}ab$, it follows that in this case the sign “ $\propto$ ” stands in the place of “ $= \frac{1}{2}$ ”, i.e. in the stead of the equals sign and the constant. This is found to be true in every case, as will be seen from succeeding formulæ.

(iii) *Circumference of a Circle.*—The circumference is always 3.1416 times the diameter, and if the Greek letter π be always put for 3.1416, the formula can be stated: $C = \pi d$ or $= 3.1416d$ where C = circumference, d = diameter, and $\pi = 3.1416$.

Note.—The longer the diameter is the longer the circumference becomes, and vice versa. Thus the circumference varies as the diameter of the circle, or $C \propto d$. But the circumference is always 3.1416 times the diameter, or $C = \pi d$. C and d are the “variables” and π is “constant”. Again we see that “ $\propto$ ”, or sign of variation, can be replaced by “ $= \pi$ ” (i.e. the sign of equality and a constant).

(iv) *The Area of a Circle.*—Let the area of the square on the radius be r^2 where r is the length of its side; r is also the radius of the circle.

Fig. 10 shows that the area of the circle is always 3.1416 times (π times) the square on the radius.

Thus $A = \pi r^2$ where A = area of a circle and r = radius. Similarly—

$$A = .7854d^2 \text{ or } \frac{\pi}{4}d^2$$

where A is the area and d the diameter of the circle.

(d) EMPLOYING THESE FORMULÆ LIKE $ab = cd$.

Example 1.—The area of a triangle is 36.75 sq. ft., its base is 18 ft. Find its height.

Method.—

$$A = \frac{1}{2}bh. \quad (\text{Dividing by } \frac{1}{2}b.)$$

$$\therefore \frac{A}{\frac{1}{2}b} = h, \text{ or } h = \frac{A}{\frac{1}{2}b} = \frac{36.75}{\frac{1}{2} \times 18} = \frac{36.75}{9} = \underline{\underline{4.08 \text{ ft.}}}$$

Example 2.—The circumference of a circle is 36.57 cm. Find its diameter.

Method.—

$$3.1416 \times 36.570 = 11.64$$

$$C = \pi d. \quad (\text{Dividing by } \pi.)$$

$$\therefore \frac{C}{\pi} = d, \text{ or } d = \frac{C}{\pi} = \frac{36.57}{3.1416}$$

$$= 11.64 \text{ cm.}$$

$$\begin{array}{r} 31 \ 416 \\ \underline{5154} \\ 3142 \\ \underline{2012} \\ 1885 \\ \underline{127} \\ 125 \end{array}$$

Note again that in arranging the formula for the sum, it is so manipulated that what you want to find appears alone on the left-hand side of the sign of equality.

Exercises 18.—Mental Exercises

1. A parallelogram is 6.5 in. long and 5 in. broad. What is its area?
2. A triangle has a base of 7 in. and an altitude of 4.5 in. What is its area? and what is the area of a rectangle of the same length and breadth?
3. What height must a triangle be, of base 4 in. long, to be equal in area to a triangle 6 in. base and 7 in. altitude?
4. A triangle has its base 4 cm. long and its perpendicular height 3.5 cm. 6 of these triangles, which are approximately equilateral, form a regular hexagon if placed side to side. What is the area of the hexagon?
5. Take $\pi = 3\frac{1}{2}$. What are the circumferences of circles which have—
 (a) Diameters measuring 2 in., 3 cm., 2.5 dm.
 (b) Radii ,, 1 in., 3 in., 2 in., 8 in., 5 in.?

6. What are the areas of squares on lines 1.5, 1.2, 3, 4 in. long? Taking π as $3\frac{1}{2}$, find the areas of the circles whose radii measure the sides of these squares.

Exercises 19

1. Find the area of a parallelogram of base 8.2 cm. whose perpendicular height is 3.7 cm.
2. A parallelogram is equal in area to a rectangle whose sides measure 6 in. and 5 in. One side of the parallelogram measures 6 in., and the angle between its adjacent sides is 30° . Construct the parallelogram, and by measurement determine the length of the other side.
3. Construct parallelograms from the following data, and in each case, by using the ruler to determine required quantities, determine their areas as accurately as possible:—

- (a) One side, 3.5 in.; second side, 3 in.; included angle, 45° .
 (b) Altitude, 2.5 in.; base, 3.5 in.; second side at 60° to the base.
 (c) One side, 8.8 cm.; perpendicular height, 6.25 cm.; angle between the given side and the base, 45° .
- Find the area of a triangle whose base is 17.59 cm. and whose altitude is 11.4 cm.
 - The area of a triangle is 506.8 sq. m., its base is 20.7 m. Find its altitude.
 - Construct a triangle having its three sides respectively 5 in., 4.7 in., and 3.2 in. Taking the longest side as its base, and by carefully measuring its altitude to the nearest $\frac{1}{16}$ in., find its area.
 - The base of a triangle is 4.9 in., and its perpendicular height is 2.5 in. It is isosceles. Construct the triangle and determine its area. Make another triangle on the same base, of equal area, and having one of the angles adjacent to the base 120° .
 - Make an equilateral triangle of 3 in. side, and make by construction a rhombus twice its area. Carefully measure the altitude of the triangle to the nearest $\frac{1}{16}$ in., and determine the area of the rhombus.
 - AB is 6.5 cm. long, the angle ABC is 35° , and the angle BAC is 57° . Using your compasses and protractor, construct the triangle. What is its altitude in mm. and its area in sq. cm.?

Exercises 20.—($\pi = 3.1416$)

- Find the circumferences of circles whose diameters are 2.35 in., 11.82 cm., 16.38 m., 1.04 ft.
- A driving wheel is 6.5 in. in diameter. How far will a point A on its circumference have travelled when the wheel has revolved 400 times? If the wheel revolves 400 times per minute, express the speed in feet per minute of a belt passing over the wheel. (*Note.*—Any point on the belt is travelling as fast as point A.)
- A driving wheel 13.5 dm. in diameter makes 750 revolutions per minute. Give the speed in metres per minute of a belt passing over and moving with this wheel.
- How many revolutions per minute would a wheel 72 in. in diameter have to make in order that any point on its circumference may travel 48 miles per hour?
- The circumference of a circular hoop is 15.6 ft. Find its diameter.
- Find the areas of circles whose diameters measure 4.3 cm., 5.6 in., and 4.38 ft.
- The radii of 3 circles measure 3.6 in., 4.7 in., and 7.35 in. Determine their areas.
- Given that a circle covers an area of 68.3 sq. m., determine its diameter and radius.

9. The circumference of a circle is 587 in. Find its area after first ascertaining its diameter.
10. Construct a square 3.5 in. side. Inscribe and circumscribe circles by drawing its diagonals. Determine the area of the circles, isosceles triangles, and square.

3. Volumes—Prisms.—In Section I certain volumes were shown to be the products of length, breadth, and thickness. The volume of the cube in fig. 5 was thus determined. There is, however, another way of regarding volume, viz. as the area of the *end* or of the *cross section* multiplied by the height or thickness of the object.

In fig. 13 the area of the shaded end is AB times AD, that is 8 squares. This area multiplied by the thickness (4 cubes thick) gives the volume as 32 cubes.

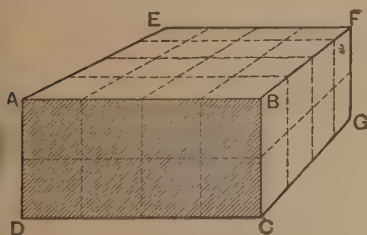


Fig. 13

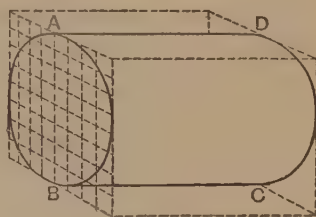


Fig. 14

Similarly, in fig. 14 the cylinder can be considered to be enclosed in a box (a rectangular prism in shape). Its volume will obviously be less than that of the prism. The area of the end AB can be found, and if multiplied by the length AD the volume is obtained.

It is necessary that an object shall be uniform if its volume is to be so determined; that is, any cross section of the object made at right angles to its height or length must be equal in size and similar in shape to its ends. Such figures are called **prisms**. (The word **cylinder** is used to mean a circular prism.) Objects other than prisms will be dealt with later. For purposes of the formula, a piece of wire may be considered to be a very long cylinder, and indeed is one if straightened.

The volume of any prism can therefore be stated thus—

$V = Ah$ where V = volume, A = area of the end or cross section, and h = the height or length.

(Note $V \propto A$ and $V \propto h$, i.e. $V \propto Ah$. Since the "constant" here is unity, $V = Ah$.)

The ends of prisms being of variable shape, the general formula can be adapted to particular cases as follows—

(a) *Rectangular Prism*.—The end is a rectangle whose $A = ab$. Thus as—

$$V = Ah \text{ and } A = ab, \therefore V = abh \dots\dots\dots 1.$$

(b) *Triangular Prism*.—The end or cross section is a triangle whose $A = \frac{1}{2}ab$. Thus as—

$$V = Ah \text{ and } A \text{ in this case} = \frac{1}{2}ab, \therefore V = \frac{1}{2}abh \dots 2.$$

(c) *Cylinder (Circular Prism)*.—The end or cross section is a circle whose $A = \pi r^2$. Thus as—

$$V = Ah \text{ and } A = \pi r^2 \text{ in this case, } \therefore V = \pi r^2 h \dots 3.$$

{ Since $A = \frac{\pi}{4}d^2$ or $.7854d^2$ also, V also equals $\frac{\pi}{4}d^2h$ or $.7854d^2h$. }

Example.—Find the volume of a cylinder whose diameter is 8 cm. and whose height is 12.5 cm. What is its weight if it is composed of brass (specific gravity or relative density of brass = 8.2)?

$$(1) V = Ah, \text{ but } A = \pi r^2 \text{ or } \frac{\pi}{4}d^2.$$

$$\therefore V = \frac{\pi}{4}d^2h.$$

Substituting given values for the variables—

$$\begin{aligned} V &= \frac{\pi}{4} \times 8^2 \times 12.5 = \frac{3.1416 \times 64 \times 12.5}{4 \times 1} \\ &= 25.1328 \times 25 = \underline{\underline{628.32 \text{ c.c.}}} \end{aligned}$$

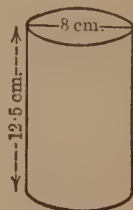


Fig. 15

(2) The specific gravity or relative density of brass is 8.2; that is, brass is 8.2 times heavier than water, bulk for bulk.

1 c.c. of water weighs 1 g.,

∴ the cylinder's volume, if composed of water,
weighs 628.32 g.

Thus the cylinder's volume in brass weighs

$$628.32 \times 8.2 = 5152.22 \text{ g. or } 5.15$$

Kg. (approx.).

$$\begin{array}{r} 628.32 \\ 8.2 \end{array}$$

$$5026.56$$

$$125.66$$

$$\underline{\underline{5152.22}}$$

Exercises 21.—Revision of Work of Section I

1. Reduce (a) 3617.8 c. in., (b) 17.28 c. in. to cubic feet.
2. How many cubic inches are there in (a) 3.7 c. ft.? (b) 14.3 c. ft.? (c) 2 c. ft. 1537 c. in.?
3. How many c.c. are there in 14 c.m. 365 c.dm. 2 c.c.?
4. Reduce (a) 1673 c.c. to c.dm., (b) 14863 c.c. to c.m.
5. What volume of water is there in a tank 3.7 m. long, 1.3 m. broad, and .75 m. deep? Give your answer in c.m. and c.dm. Given that a litre capacity is 1 c.dm., how many litre bottles could be filled with the water in the tank?
6. If the tank of question 5 were filled with oil (sp. gr. .75, i.e. .75 the weight of water, bulk for bulk) instead of water, find the weight of the oil.
7. A c.dm. of water weighs 1 Kg. Find the weight of water in a tank 3 m. long, 2 m. broad, and 1.3 m. deep.
8. Find the volume of a rectangular prism whose dimensions are 3.6 ft. long, 2.5 ft. broad, and 1.75 ft. high.
9. A cubic foot of water weighs 1000 oz. Find the weight of water in a tank whose internal dimensions are 6 ft. by 5.4 ft. by 1.35 ft., if the tank is two-thirds full.

Exercises 22

1. Find the volume of a rectangular prism 5 ft. 3 in. long, 4 ft. 8 in. broad, and 2.75 ft. thick.
2. Find the volume of a cylinder whose diameter is 6.8 in. and whose length is 1.5 ft.
3. Find the volume of a triangular prism, given that the base and altitude of the triangular cross section are 7.35 cm. and 8.36 cm. respectively, and the length of the prism is 14.3 cm.
4. A tank holds 3 c.m. of water, and it is quite full. The tank is 15 dm. long and 10 dm. broad. What is its depth?
5. The volume of a cylinder is 22 c. ft.; it is 17.6 ft. high. Find the area of its cross section and its diameter.
6. The altitude of an equilateral triangle is always .866 times its base. The volume of a triangular prism whose cross section is an equilateral triangle of 6 in. side is 15,588 c. in. Find its height.
7. A cistern whose external measurements are 10 ft. long, 7 ft. broad, (C 200)

and 3 ft. deep, is made of slate slabs 1.5 in. thick. Find the volume of water it will hold, and the weight of water in it when half-full.

8. The area of one face of a cube is 289 sq. in. Find the volume of the cube and the length of one edge. If it be made of material 1 c. in. of which weighs 3.25 oz., find its weight.
9. Find the length of the edge and the area of one face of cubes whose volumes are (a) 343 sq. in., (b) 8000 c.c.
10. Calculate the volumes of the following prisms from the data given:—
 - (a) Height, 16.7 in.; base, a square whose sides are 3.5 in.
 - (b) Height, 14.37 in.; base, a triangle of base 3.76 in. and altitude 4.38 in.
 - (c) Height, 16.8 in.; base, a right-angled triangle, the sides containing the right angle being 3.6 in. and 4.75 in. respectively.
11. Work the following exercises, based on cylinders:—
 - (a) Volume, 560 c. in.; area of end, 12.5 sq. in. Find height.
 - (b) Volume, 6375 c.c.; area of cross section, .725 sq. cm. Find length.
 - (c) Volume, 201.0625 c. in.; height, 4 ft. Find the diameter of the cross section.
12. In a cylindrical steel plate 4 ft. 6 in. diameter and 1.5 in. thick twenty holes are drilled for bolts, each being 1.25 in. in diameter. Find the volume of steel in the finished plate and its weight if 1 c. in. of steel weighs 0.29 lb.

The **specific gravity** or **relative density** of any substance is a number expressing how many times it is heavier than water, bulk for bulk.

SOME USEFUL FACTS

Substance.	Weight of 1 c. in.	Relative Density or Specific Gravity.		
Water	.036 lb.	weighs 1.	{ times that of an equal vol. of water.	
Cast iron.....	.26	"	7.1	"
Steel.....	.29	"	7.85	"
Brass.....	.3	"	8.4	"
Copper.....	.32	"	8.9	"
Lead.....	.412	"	11.4	"
Alcohol.....	.029	"	.8	"
Cork.....	.009	"	.25	"
Mercury	.484	"	13.6	"
Wood (ash).....	.027	"	.75	"

Note.—If 1 c.c. of water weighs 1 g., then 1 c.c. of steel weighs 7.89 g., 1 c.c. of lead 11.4 g.; and so on. Similarly, if 1 c. ft. of water weighs 1000 oz., then 1 c. ft. of copper weighs 8900 oz., 1 c. ft. of cast iron 7100 oz.; and so on.

If 1 c. in. of water weighs .036 lb., and copper is 8.9 times the weight of water, bulk for bulk, then 1 c. in. of copper weighs $.036 \times 8.9 = .32$ lb.

Exercises 23

(Use the Table given above)

1. Find the weights of cubes of the following substances:—(a) a cube of copper of 1.75 in. edge, (b) cube of steel of 3.68 in. edge.
 2. A cube of steel weighs 18.56 lb. Find the volume of the cube.
 3. A cylinder of copper weighs 133.5 g. Find its volume.
 4. A coil of copper wire weighs 16 lb. Find its volume. If the length of the wire is 20 ft., find the area of its cross section.
 5. How many bullets each .01 c. in. can be made from 1 lb. of lead?
 6. 4 oz. of copper are melted with 1 oz. of lead. What is the volume of the alloy?
 7. A column of mercury 30 c. in. in volume is supported by the pressure of the atmosphere on a dry day. Find the pressure of the atmosphere if the cross section of the column of mercury is 1 sq. in.
4. Many statements can be reduced to equations in the form $ab = cd$. E.g. (a) The strength of an electric current in amperes is equal to the voltage divided by the resistance of the circuit in ohms. Find the total resistance present when a force of 15 volts is required to send a current of 1.5 amperes through a certain circuit.

The equation (formula) can be made up from the first statement, if C = amperes, E = volts, and R = ohms resistance.

$$\text{Then } C = \frac{E}{R}, \text{ or } CR = E.$$

$$\therefore R = \frac{E}{C} = \frac{15 \text{ volts}}{1.5 \text{ amp.}} = 10 \text{ ohms resistance.}$$

(b) The distance in feet a body falls from rest is equal to half the acceleration due to gravity multiplied by the square of the time in seconds during which it has been falling. Find the time a body has taken to fall from a tower 80 feet high, if the acceleration due to gravity is 32 ft. per second.

If s = distance in feet, g = acceleration due to gravity, and t = time in seconds, it follows from sentence one of the question that—

$$s = \frac{1}{2}gt^2,$$

$$\text{i.e. } t^2 = \frac{s}{\frac{1}{2}g} = \frac{80 \text{ ft.}}{\frac{1}{2} \times 32 \text{ ft. per sec.}} = 5;$$

$$\text{but if } t^2 = 5, \text{ then } t = \sqrt{5} = 2.23 \text{ sec.}$$

The body has therefore been travelling just under $2\frac{1}{4}$ sec.

Exercises 24

1. Write down in algebraic form—

- (a) The product of a and x^2 divided by the cube of y .
- (b) Multiply the cube of c by the square of x and extract the square root.
- (c) The product of a and b divided by m is equal to three times the product of x and y .

2. Express in algebraic form—

- (a) The cube root of the product of a , b , and c .
- (b) The square of a is equal to the product of b and c .
- (c) The fourth power of x multiplied by y is equal to 15 times the square root of y divided by x .

3. The resistance of a wire is given by the formula $aR = lK$. Find R , given that $l = 100$, $a = .002$, and $K = .00002117$.
4. $746 \text{ HP} = \text{CV}$. This equation expresses the relationship between horse-power (HP), electric current in amperes, and electro-motive force in volts. Given $V = 220$ volts, and $C = 40$ amp., find HP.
5. The force (F) exerted between two magnetic poles is equal to the product of their strengths (a and b) divided by the square of the distance (d^2) between them. Express the necessary formula, and given that $a = 6$, $b = 8$, and $d = 4$ cm., find F .
6. The heating effect (H) of an electric current is given in terms of amperes, ohms, and seconds by the equation—

$$H = .24C^2Rt.$$

Find the current C in amperes when $H = 765$ calories, $t = 240$ sec., and $R = 12$ ohms.

7. y equals the square root of x . Find x when $y = 3.5$.
8. The distance (S) in feet travelled by a body in T seconds is given by $S = 16T^2$. Find S when $T = 3.5$ sec.
9. The rate (V) at which a body is moving after travelling a certain

time is given by the formula $V = ft$. If $t = 18.5$ sec., and $f = 13.65$ ft. per second, find V , the rate of motion.

10. The volume (V) of a sphere is given by the equation $3V = 4\pi r^3$, where r is its radius. If $r = 4.75$ cm., find V .

11. If $w = k\frac{bd^2}{l}$, find k if $w = 540$, $b = 1$, $d = 2$, and $l = 10$.

12. The area of a triangle in terms of its three sides, a , b , and c , is given by the formula $A = \sqrt{s(s-a)(s-b)(s-c)}$, where s equals half the sum of the sides a , b , and c . The three sides of a triangle are 10, 12, and 16 in. respectively. Find its area.

SUMMARY

If $ab = cd$, then $a = \frac{cd}{b}$, and $\frac{a}{d} = \frac{c}{b}$; or—

a multiplier transferred across the sign of equality becomes a divisor. Similarly, a divisor becomes a multiplier when carried to the other side of an equation.

Substitution.—By substituting known values for any three of the *terms* of the expression $ab = cd$, the fourth term can be determined.

Mensuration Formulæ.—Like $ab = cd$.

Rectangle.—Area equals length multiplied by breadth, or $A = lb$. Thus—

$$l = \frac{A}{b}, \text{ and } b = \frac{A}{l}$$

The area of the four walls of a room is the height of the room multiplied by its perimeter (or sum of the lengths of the walls). They form a rectangle if placed in one plane.

Parallelograms on equal bases and of equal altitude are equal in area. They are thus equal to rectangles of equal base and altitude, and their areas equal their lengths multiplied by their *perpendicular* heights.

Triangles are half the area of parallelograms, on equal bases and of the same height. The area therefore equals half the base multiplied by the height, or $A = \frac{1}{2}bh$. Thus—

$$b = \frac{A}{\frac{1}{2}h}, \text{ and } h = \frac{A}{\frac{1}{2}b}$$

Circle.—The length of the circumference is always π times the length of the diameter of a circle, i.e.—

$$C = \pi d, \text{ or } C = 2\pi r. \quad (\pi = 3.1416.)$$

The area is always π times the area of the square on the radius, or $\frac{\pi}{4}$ times the area of the square on the diameter, i.e.—

$$A = \pi r^2, \text{ or } A = \frac{\pi}{4} d^2. \quad \left(\frac{\pi}{4} = .7854.\right)$$

Prisms.—The volume of a prism is the product of the area of its cross section and its length.

Rectangular Prism.—

$$V = Ah, \text{ i.e. } V = lbh. \quad (\text{A of rectangle} = lb.)$$

Triangular Prism.—

$$V = Ah, \text{ i.e. } V = \frac{1}{2}abh. \quad (\text{A of triangle} = \frac{1}{2}ab.)$$

Cylinder.—

$$V = Ah, \text{ i.e. } V = \pi r^2 h. \quad (\text{A of circle} = \pi r^2.)$$

The **specific gravity** or **relative density** of a body is the number of times it is heavier than water, volume for volume. By means of it, volumes can be obtained from weights, and vice versa. For this purpose the statements that 1 c.c. of water weighs 1 g., and 1 c. ft. of water weighs 1000 oz., form the bases of reckoning.

Many **laws** can be expressed by formulæ in the form $ab = cd$. Among them are (a) Ohm's law ($C = \frac{E}{R}$); (b) law of heating effect of electric current ($H = C^2 R t$); (c) law for determining distances travelled by falling bodies from rest ($s = \frac{1}{2}ft^2$), &c.

Incidentally mentioned are formulæ for—

(1) **Sphere.**— $V = \frac{4}{3}\pi r^3$.

(2) **Triangle.**—Given the lengths of the three sides—

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$

(s = half sum of sides a , b , and c).

SECTION III.—RELATIONSHIPS

(OR THE EQUATION $\frac{a}{b} = \frac{c}{d}$)

In the foregoing sections *comparisons* have been made. E.g. (1) Lengths and areas have been considered as so many times the units of length and area respectively; (2) the circumference and area of a circle have been found to be π times the diameter and the square on the radius respectively.

There are, then, *relationships* existing between the circumference of the circle and its diameter, between the length to be measured and the yard measure you are using to determine that length; and so on. Such relationships are expressed by *ratios*.

A **ratio** is the *relationship* between magnitudes of the same kind, and is expressed in the form $\frac{a}{b}$ or $a : b$, where a and b are the two magnitudes.

Example 1.—One field is 3 acres and another 5 acres in extent.

Field A is $\frac{3}{5}$ field B, or field B is $\frac{5}{3}$, i.e. $1\frac{2}{3}$, times field A.

The relationship of field B to field A in terms of field A is therefore $\frac{5}{3}$, which means that taking field A as the standard of comparison, field B is $\frac{5}{3}$ or $1\frac{2}{3}$ times its area.

Expressed in terms of field B, field A is $\frac{3}{5}$ its size.

$\frac{5}{3}$ and $\frac{3}{5}$ are termed *ratios*, since they show this relationship.

Example 2.—One line is 10 m. long, another 2.5 m. It is obvious that the longer is four times the shorter, or $\frac{10}{2.5}$, which is 4.

Thus expressed in terms of the shorter line the longer is four times its length, and, vice versa, the shorter line is $\frac{1}{4}$ of the longer one.

The ratio of 2.5 m. to 10 m. is therefore $\frac{2.5}{10}$ or $\frac{1}{4}$, and the ratio of 10 m. to 2.5 m. is $\frac{10}{2.5}$ or $\frac{4}{1}$.

Note.—The ratio of 6 in. to 7 in. may be expressed as $\frac{6}{7}$ or as 6 : 7. The latter form is not necessary, and for purposes of this course ratio will be expressed in the more convenient form $\frac{6}{7}$.

Exercises 25

1. Express 6 in. as the ratio of 16 in.
2. What is the ratio of 9 in. to 1 yd., and of a ft. to x ft.?
3. What sums of money bear to £1 the following ratios: $\frac{1}{8}$, $\frac{3}{8}$, $\frac{2}{3}$, $\frac{7}{8}$, $\frac{1}{2}$?
4. What sum will bear the same ratio to 5s. that 5 lb. do to 60 lb.?
5. Find the relationship between 1 ft. 6 in. and 2 ft.
6. If a triangle of altitude 6 in. is 146·3 sq. in. in area, then a triangle of base equal to the former but of altitude 18 in. has an area of 438·9 sq. in. Prove the correctness of the statement by formula, and express the ratios of the areas and the altitudes.
7. Draw AB 2 in. long. At A make a right angle, and on the perpendicular arm mark off distances AC and AD which are respectively 1·5 in. and 3 in. Join BC and BD. You have now two triangles, ABC and ABD, whose altitudes and areas vary.
 - (a) What is the ratio of the altitudes?
 - (b) Determine the areas of the triangles, and express their relationship as a ratio.
 - (c) Let AC be a units long and AD be b units long. What is the ratio of AC to AD?
 - (d) If the areas of the triangles be c and d sq. in. respectively, express their ratio.
 - (e) From your observations made in (a) and (b) you can write down $\frac{a}{b} = \frac{c}{d}$. Explain this statement.
8. Construct an angle BAC 60° . Let AB be 2 in. and AC be 2·5 in. Join BC. Produce AB and AC to D and E. Make AD 4 in. long, and from D draw DE parallel to BC to meet AC in E.
 - (a) What are the measures to the nearest ·1 in. of BC, AE, and DE?
 - (b) Express the ratio of AB to AD, AC to AE, BC to DE.
 - (c) What do you note concerning all these ratios if expressed in their simplest form?
 - (d) Calling the three sides of each triangle a , b , c , and d , e , f respectively, what do you understand by the statement $\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$?
9. Express the ratio of a to b , of a^2b to abc , of x^2m to x^3y , of $x + d$ to $x + 2d$.
10. If the ratio of a to b is equal to the ratio of c to d , and a is 6 when b is 4 and d is 8, find c .
11. Find the relationship existing between the areas of two circles whose radii are respectively 4 and 6 in.
12. A cylinder has a diameter of 4·5 in. and a length of 20 in. It is proposed to divide it into two parts bearing the relationship one to the other of $\frac{3}{7}$. What is the length and volume of each portion?

13. Two squares have their sides 5 in. and 7 in., respectively. State the ratio of their areas.
14. What number bears the same ratio to 6.5 that 15.7 does to .314?

THE COMPARISON OF RATIOS

Many problems depend on the careful statement of ratios, of which three types are presented in the following data:—

1. *Data*.—If 8 articles cost 10s., then 12 articles will cost 15s. at the same rate.

The ratio of the articles $= \frac{8}{12}$, which, reduced to lowest terms, $= \frac{2}{3}$.

The ratio of the prices $= \frac{10}{15}$, which, reduced to lowest terms, $= \frac{2}{3}$.

You note that the ratio of the articles equals the ratio of the prices **directly**. (Direct proportion.)

2. *Data*.—If 7 men do a piece of work in 3 d., then 1 man working alone should do the work in 21 d.

Ratio of the men $= \frac{7}{1}$.

Ratio of the days in the same order, $= \frac{3}{21}$, which, reduced to the lowest terms, $= \frac{1}{7}$.

You note here that the ratio of the men equals the ratio of the days, **if inverted**. (Inverse proportion.)

3. *Data*.—If the radius of a circle is 2 in., its area is 12.5664 sq. in.; and if 3 in., its area is 28.2744 in.

The ratio of the radii $= \frac{2}{3}$.

The ratio of the areas $= \frac{12.5664}{28.2744}$, which, reduced to lowest terms, $= \frac{4}{9}$.

You note here that to make the ratio of the radii equal the ratio of the areas it will be necessary to *square* the former. In other examples of this type other operations will be necessary in order to express the equality of the ratios, and special methods are employed to deal with such problems.

These three types of problems show that if $\frac{a}{b}$ is one ratio and $\frac{c}{d}$ is the other ratio, then either—

(1) $\frac{a}{b} = \frac{c}{d}$ when $\frac{a}{b}$ is said to be *directly proportional* to $\frac{c}{d}$; or

(2) $\frac{a}{b} = \frac{d}{c}$, where $\frac{c}{d}$ has to be *inverted* to equal $\frac{a}{b}$, when $\frac{a}{b}$ is said to be *inversely proportional* to $\frac{c}{d}$; or

(3) $\frac{a}{b}$ has to undergo some further operation to make it equal to $\frac{c}{d}$ and the proportion idea in its simple form is not readily seen.

Exercises 26

Work the following exercises from the given data:—

1. When the diameter of a circle is 10 in., its circumference is 31.416 in.; when the diameter is 8 in., the circumference is 25.1328 in. Express the ratio of the diameters and the ratio of the circumferences. What do you note when the ratios are reduced to lowest terms? Is this a case of *direct* or *inverse proportion*?
2. A pole 6 ft. high casts a shadow 9 ft. long, and a lamp-post 10 ft. high casts a shadow of 15 ft. at the same instant. Express the ratios of the heights and of the lengths of shadow in the lowest terms. Is this *direct* or *inverse proportion*?
3. 2 men do a piece of work in 6 days; 3 men do it in 4 days. What is the ratio of the men and the ratio of the days? What must be done to make these ratios equal?
4. A train runs 150 m. in 5 h. and 90 m. in 3 h. Express the ratios of the miles and hours, and the equation connecting them.
5. Two triangles of equal area have their bases 5 in. and 7 in., and their altitudes 3.5 in. and 2.5 in. respectively. What is the ratio of their bases and the ratio of their altitudes? Is this a case of *direct* or *inverse proportion*? Give reasons for your answer.
6. The sides of two squares are 4 in. and 2.5 in., and their areas are respectively 16 sq. in. and 6.25 sq. in. Express the ratios of the sides and of the areas in their lowest terms, and say how one ratio must be treated to make it equal the other.
7. A stone dropped from a tower is found to fall 64 ft. in 2 sec. and 144 ft. in 3 sec. Compare the ratios of the times and distances through which it falls.
8. The length of the arm of a lever to which a weight of 4 lb. is attached is found to be 8 in., but this arm has to be increased to 16 in. if the weight attached is reduced to 2 lb. Express the ratios of the weights and the arms, and state the relationship existing between these ratios.
9. An insurance company shows the following scale for the purchase of a life policy:—

Age.		Annual Premium.
26		£5 3 4
30		7 5 0

State the ratios of the ages and the premiums. Is there any visible relationship existing between these ratios?

From this exercise you have become accustomed to stating the ratios of certain given data, and of expressing the relationship existing between these ratios. In the succeeding exercises the same processes will be employed, but with this difference, that while in the above data all four terms making up the two related ratios are given, in the following problems one term will be wanting, and must be supplied by x . This missing term will be in every case the answer of the problem.

DIRECT SIMPLE PROPORTION—EQUAL RELATIONSHIPS

$\frac{a}{b}$ is a ratio and expresses the relationship of a things to b things of the same kind. Similarly, $\frac{c}{d}$ expresses the relationship of c things to d things of the same kind.

If a bears the same relationship to b that c does to d then $\frac{a}{b} = \frac{c}{d}$ and the four terms are said to be *proportional*.

By cross multiplication, or by multiplying both sides of the equation by bd , $\frac{a}{b} = \frac{c}{d}$ becomes $ad = bc$ —the equation of Section II.

Simple proportion is the *equality* of *ratios*, and the method of working such sums only involves a knowledge of the equation $\frac{a}{b} = \frac{c}{d}$ after the ratios of the quantities have been expressed.

Example.—If 70 articles cost £8, 16s., what will 84 articles cost?

Method.—Every proportion sum contains at least two sets of terms. In this one they are articles and cash.

$$\begin{array}{rcl}
 70 \text{ articles cost } & \text{£}8, 16s. & \\
 84 \quad \quad \quad & \text{,,} \quad \quad \text{,,} & \text{£}x. \\
 \text{Ratio of articles } & \frac{70}{84} & \text{Ratio of cash } \frac{\text{£}8, 16s.}{\text{£}x}.
 \end{array}$$

Since the same relationship must exist between the articles as exists between the amounts of cash.

$$\frac{70}{84} = \frac{\text{£}8, 16s.}{x},$$

$$\text{i.e. } 70x = 84 \times \text{£}8.8,$$

$$x = \frac{12}{\cancel{84}} \times \frac{\text{£}8.8}{70} = \frac{\text{£}10, 11s. 2\frac{1}{4}d.}{10}$$

Exercises 27

- Given that $\frac{a}{b} = \frac{c}{d}$, and that $a = 8.7$, $b = 6.43$, and $c = 10.82$, find d .
- A train travels x m. in a h., and travelling at the same rate covers y m. in b h. Express the relationships existing among x , y , a , and b . If $x = 350$, $a = 7$ h. 30 m., and $b = 6$ h. 5 m., find y .
- Complete the following proportion sums, determining in each case the value of x :—

$$(a) \frac{6.4}{7.3} = \frac{x}{4.65}, (b) \frac{b^2}{cd} = \frac{cd}{x}, (c) \frac{3x}{4.7} = \frac{12}{7.43}.$$

- If 1 t. 4 cwt. of coal can be purchased for 19s. 6d., find the cost of 6 t. 17 cwt.
- A bankrupt's liabilities amount to £4365, and his assets realize £2910, what can he pay in the £1?
- A man borrows £350, for which he agrees to pay £14 per annum interest, what amount of interest should he pay for a loan of £475? .
- The carriage of 15 t. of metal for 40 m. is 3 g.; what should be charged for the carriage of 23 t. (a) the same distance, (b) 20 m.?
- A train goes $3x$ m. in 4 h.; how far will it go in $\frac{x}{4}$ h. at the same rate? If $x = 40$, find the distance gone in miles.
- I buy x oranges for 5s. and y oranges for 4s. 3d. Assuming that they are bought at the same rate, what is the relationship existing between the two quantities of oranges? If $x = 50$ oranges, find y .
- A train is travelling at the rate of x m. per hour., how long will it take to travel b m.? If $x = 30$ and $b = 180$, express your answer numerically.
- If the cost of paving a yard 14 ft. by 10 ft. is £3, find the cost of paving a yard 3 ft. 8 in. longer and broader at the same rate.
- If $\frac{7}{8}$ lb. cost $1\frac{3}{8}$ s., what will be the cost of $1\frac{7}{12}$ cwt.?

Similar Triangles.—If in two triangles the angles of one triangle are equal to those of the other, each to each, the triangles are said to be *similar*.

In fig. 16, the triangles ABC and DEF are *similar* if $\angle B = \angle E$, $\angle A = \angle D$, and $\angle C = \angle F$.

If the triangle ABC be placed upon the triangle DEF so that $\angle B$ falls on $\angle E$, then the sides c and a will fall on f and d

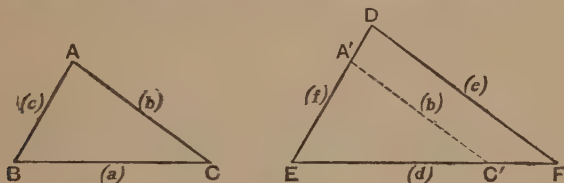


Fig. 16

respectively, and b will be parallel to e . This figure is like the one to be constructed in Ex. 24, No. 8, and it was found that the ratios of the similarly placed sides were equal to one another. So in this case the side DE is just as many times longer than the side AB as the side EF is longer than the side BC, and DF than AC, or—

$$\frac{c}{f} = \frac{a}{d} = \frac{b}{e}$$

This is what is best expressed by the statement that *the sides of similar triangles about equal angles are proportional*.

Note.—To Letter a Triangle.—Roman capitals stand at the angular corners, and $\angle A$ means “the angle BAC”. The sides subtending the angles A, B, and C are denoted by small letters a , b , c . Side a subtends $\angle A$; b subtends $\angle B$; and so on.

If this lettering be adopted as a satisfactory method of speaking of a triangle, there will be no difficulty in understanding what is meant by “ $\angle A = 60^\circ$ ”, or “ a has 10 units of length”, &c.

Exercises 28

1. Construct a triangle ABC, having its three sides, a , b , and c , respectively 2, 2.5, and 3 in. Produce BC (that is a) to F, making BF measure 4 in. Through F draw a line parallel to AC (that is b) to meet AB (or c) produced. You have now two superimposed

similar triangles. Why do the sides of the second triangle respectively measure 4, 5, and 6 in.?

2. Two similar triangles ABC and DEF are constructed. If the sides of the triangle ABC are 5.4 in., 3.2 in., and 4.7 in. long respectively, and the side of the triangle DEF corresponding to the 5.4 in. side of ABC is 2.7 in., what are the lengths of its remaining sides?
3. XYZ and ABC are similar triangles. XY is 10.2 cm. and XZ is 5.3 cm. long. If AB is 6.7 in., find the length of AC.
4. By constructing similar triangles with their two similarly placed sides respectively 2.5 cm., 4.5 cm., and 7.25 cm., x cm., determine x by measurement and by the statement $\frac{2.5}{7.25} = \frac{4.5}{x}$.
5. Construct a right-angled triangle ABC, of which $\angle A$ is the right angle and $\angle B$ is 60° . Let AB be 3 in. long. Produce it to D, making AD 4 in. From D draw a line parallel to BC, making ADE similar to ABC.

(a) What are the exact measures of a , b , and c ?

(b) Determine the length of DE and AE by using the process of simple proportion, and check your answers by measurement.

6. ABC and ADE are two similar triangles, the $\angle A$ being 30° . AB is 3 in. and AD is 5 in. Given that $\angle C$ and $\angle E$ are right angles, construct the similar triangles, and determine the lengths of their sides both by measurement and by using the equation $\frac{a}{b} = \frac{c}{d}$.

Similar triangles may be used to *graphically represent* simple proportion sums. The similarly placed sides are made to represent the terms of the proportion by adopting a suitable scale, and the unknown term can then be measured off. To explain this a very simple example is given.

Example.—If 6 books cost 7s. 6d., what will 8 books cost at the same rate?

Analysis of sum—The ratios are between “books” and “cash”.

$$\text{Obviously ratio of “books”} = \frac{6}{8}$$

$$\text{and ratio of “cash”} = \frac{7s. 6d.}{x} \quad \text{Also } \frac{6}{8} = \frac{7s. 6d.}{x}$$

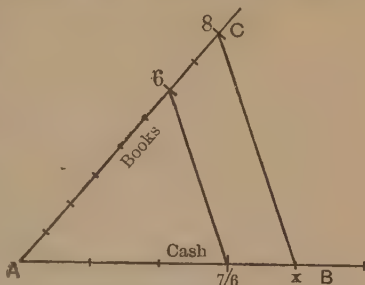
METHOD OF GRAPHICAL REPRESENTATION.—1. Take any angle BAC, and agree that one arm (say AC) shall represent

"books" to a convenient scale (say 1 cm. = 1 book), and the other arm shall represent "cash" to a convenient scale (say 2 cm. = 1 half-crown).

2. Mark off 6 cm. and 8 cm. along AC to represent 6 and 8 books and intervals of 2 cm. along AB to represent half-crowns, marking the third as 7s. 6d. (i.e. three half-crowns).

3. Now since 6 books cost 7s. 6d., join the "6 books" mark with the "7s. 6d." mark.

4. From the "8 books" mark draw a line parallel to this one, and it will meet AB at a point x , which will give the required answer according



to the cash scale agreed on. For since the triangles are similar the lengths of their similarly placed sides are proportional, i.e.—

$$\frac{6 \text{ cm.}}{8 \text{ cm.}} = \frac{3 \times 2 \text{ cm.}}{x \times 2 \text{ cm.}} \text{ or } \frac{6 \text{ books}}{8 \text{ books}} = \frac{3 \text{ half-crowns}}{x \text{ half-crowns}}$$

It is easy to see that $x = 4$ half-crowns or 10s.

Exercise 29

By using similar triangles, work the following proportion sums:—

1. If 10 yd. of material cost 30s., what is the cost of 15 yd.?
2. If I get £5 interest on a £100 loan, what must I loan to get £8, 15s.?
3. If I invest £80 to get an income of £2½, what income do I get on every £100 invested?
4. If 14 c. in. of metal weigh 1.5 lb., what will 36 c. in. weigh?
5. The price of 8 cows is equal to that of 13 sheep. How many cows could be exchanged for 65 sheep?
6. 30 ft. of wire weigh 2 oz. Find the weight of a coil of 165 ft. of wire.

Percentages.—"Per cent" means "per hundred"; the hundred being a convenient basis for the comparison of magnitudes. For example a man can get £29¼ for the loan of £650; he can also get, in another quarter, a return of 5s. for every £5 he lends. He wishes to know which is the better financial arrangement. He therefore determines how much "per cent" he can get for

his money, i.e. how much money he can get for every £100 he lends. To do this he has recourse to the equation $\frac{a}{b} = \frac{c}{d}$.

In Case 1, he gets £29 $\frac{1}{4}$ for every £650, i.e. he gets £ x for every £100.

$$\therefore \frac{29\frac{1}{4}}{x} = \frac{650}{100} \quad x = 4.5. \quad (4.5 \%)$$

In Case 2, he gets 5s. for every £5, i.e. he gets £ x for every £100.

$$\therefore \frac{\text{£}1}{\text{£}x} = \frac{5}{100} \quad x = 5. \quad (5 \%)$$

The second is therefore the better financial arrangement, since he can get "5 per cent" (written 5 %) for his money, whereas in the former he obtains only 4.5 %.

Note.—5 % means £5 of £100, 5s. of 100s. or £5, 5*d.* of 100*d.* or 8s. 4*d.* The figure 5 is neither £5, 5s., or 5*d.* A 1 % increase in weight means that 100 g. has been increased to 101 g., 100 oz. has been increased to 101 oz., and so on; while a decrease of 2 % in volume means that a litre of the substance has become 980 c.c., and 100 pt. now measure 98 pt.

Exercises 30

1. What is 5 % of £1, £18, £7, 10s., £3475?
2. Of what amounts are the following 5 %:—1s., 6s., 10s. 6*d.*, 23s. 4*d.*, 16 g., 4 l., 7 oz.?
3. What is 2 $\frac{1}{2}$ % of each of the following:—£1; £7, 15s.; £43, 10s.; £3, 6s. 8*d.*?
4. What is the brokerage on £4360, 12s. at $\frac{1}{8}$ %?
5. Find the commission to be paid on £736, 7s. 6*d.* at 2 $\frac{1}{2}$ %.
6. 60 % of a class of 60 scholars passed an examination. How many failed?
7. What interest must be paid on £638, 12s. 6*d.* at 5 % per annum? What will this interest amount to in 4 years?
8. A bronze consists, say, of 91 % copper and 9 % tin. How many cubic inches of copper and tin respectively are required to make a cubic foot of bronze?
9. If eggs are bought at $\frac{3}{4}$ *d.* and sold at 1*d.* each, find the percentage profit made.
10. A collector received £3, 6s. 8*d.* as commission. This represents 2 $\frac{1}{2}$ % of the debts collected. What did the debts amount to?

11. By selling coal at 17s. per ton a man loses 3%. Does he gain or lose by selling it at £1 per ton, and by how much per cent?
12. After paying an income tax of $2\frac{1}{2}\%$ a man finds his net income to be £975. Find his gross income.
13. An alloy of copper and tin is made up of 1 lb. of copper to every 1 oz. of tin. Find the percentage of copper and tin in the alloy.
14. A man's gross income is £350. After paying income tax he receives £330 as his net income. Find the percentage paid in taxes.
15. 23% of the value of a man's property has been destroyed by fire. His remaining property is valued at £810. Find the value of the property lost and the total value of the property before the fire.
16. A sells a horse to B at a gain of 5%, but afterwards buys him back, so that B gains 4% on his money, in order to sell the horse to C. If the original value of the horse was £20, for what must A sell it to C to neither gain nor lose by the transactions? How did A stand if C paid £25 for the horse?
17. An iron bar expands .04% when heated by the sun of summer. If a railway line measures 32 ft. at freezing-point, what length will it be when heated to this extent in summer?
18. A, B, and C are jointly interested in a vessel, A to the extent of 40%, B 25%, and C 35%. The profits of a certain voyage amount to £450. How should this amount be shared among A, B, C?

Simple Interest.—Money (**Principal**) is lent out at a certain **Rate** per cent per annum, and in a given **Time** realizes a certain **Interest** or amounts to a given **Sum**. Thus £300 (**Principal**) lent at the **Rate** of 4% per annum realizes in one year £12 **Interest**, and amounts to £312. In 3 y. it would realize an interest of £36, and would amount to £336.

Example 1.—Find the simple interest of £560 for 3 y. at 4% per annum. *Method*—

£100 invested for one year yields £4 interest.

∴ £560 „ „ £x „

Ratio of principals $\frac{100}{560}$. Ratio of interests $\frac{4}{x}$.

Then $\frac{100}{560} = \frac{4}{x}$, or $x = \frac{£560 \times 4}{100}$ in one year.

Thus in 3 y., $x = \frac{£560 \times 4 \times 3}{100} = \underline{\underline{£67, 4s.}}$

Substituting P for the Principal, £560; R for the Rate,

4 %; and T for the time, 3 y.; the interest, £ x or I, can be written—

$$I = \frac{P \times R \times T}{100}.$$

This formula is the one usually adopted for determining simple interest, and is based upon the equation $\frac{a}{b} = \frac{c}{d}$ as shown above.

This formula can be obtained directly. Thus—

$$\begin{array}{lcl} \text{£100 invested yields £R per annum.} \\ \therefore \text{£100} & \text{,,} & \text{,,} \quad \text{£R} \times T \text{ in } T \text{ years.} \\ \text{But £P} & \text{,,} & \text{,,} \quad \text{£}x \quad \text{,,} \quad \text{,,} \\ \text{i.e. } \frac{100}{P} & = & \frac{R \times T}{x}, \\ \text{or } 100x, \text{ i.e. } 100 I & = & P \times R \times T. \end{array}$$

Thus 100 times the Interest equals the product of Principal, Rate, and Time.

The formula can be manipulated like the equation $ab = cd$.

$$\text{If } 100 \times I = P \times R \times T.$$

$$\text{Then } I = \frac{P \times R \times T}{100}.$$

I.e. the Interest is found by dividing the product of Principal, Rate, and Time by 100. Also—

$$\frac{100 \times I}{P \times R} = T, \quad \frac{100 \times I}{P \times T} = R, \quad \frac{100 \times I}{R \times T} = P.$$

Expressed verbally—

The **Time** of a loan is found by dividing 100 times the Interest by the product of Principal and Rate;

The **Rate** at which a sum is lent is found by dividing 100 times the Interest by the product of the Principal and Rate; and

The **Principal** lent is 100 times the Interest divided by the product of the Rate and Time.

Example 2.—What sum of money must be lent to produce a simple interest of £15 in 3 y. at 4 % per annum.

Here $I = £15$, $T = 3$ y., $R = 4\%$.

Given $100 I = P \times R \times T$.

$$\text{I.e. } 100 \times 15 = P \times 4 \times 3. \quad \therefore P = \frac{100 \times 15}{4 \times 3} = £125.$$

Example 3.—In what time will £85 amount to £90 at 7% per annum.

Here $I = £5$, $P = £85$, $R = 7$.

Given $100 \times I = P \times R \times T$.

$$T = \frac{100 \times I}{P \times R} = \frac{100 \times 5}{85 \times 7} = \frac{100}{119} \text{ y., or } 307 \text{ d. (approx.).}$$

Example 4.—What sum of money will amount to £450 in 2 y. at 4% per annum. A special method is required here because the interest is not stated.

Method—

£100 loaned realizes £8 interest in 2 y. at 4% per annum.

$\therefore$ £100 amounts to £108 in 2 y. at 4% per annum.

$$\begin{array}{ccccccc} \text{£}x & & \text{£}450 & & & & \\ \text{I.e. } \frac{100}{x} & = & \frac{108}{450} & , & x = & \frac{45000}{108} & = \underline{\underline{\text{£}416, 13s. 4d.}} \end{array}$$

Exercises 31

- Find the simple interest on £350 for 4 y. at 2.5% per annum.
- A man banks to-day £175. What sum of money can he withdraw in $3\frac{1}{2}$ y. at $3\frac{1}{4}\%$ per annum?
- I borrow £435. What interest must I pay at $2\frac{1}{2}\%$ per annum if I return the loan after 146 d.?
- By investing £375 for 4 y. I get £60 interest. What is the rate of interest per cent per annum?
- In 1875 the population of a town was 42,000, in 1900 it was 54,350. What was the increase per cent per annum?
- A money lender charged £48, 10s. for the loan of £840 at $2\frac{1}{2}\%$ per annum, simple interest. For how long did he lend it?
- I invested £320 at $4\frac{1}{2}\%$ per annum. After a certain time I was able to withdraw £465. How long was the sum invested at simple interest?
- In $7\frac{1}{2}$ y. £340 amounted to £442. What rate of interest per annum was paid on the money?
- How long must £760.5 be invested to yield an interest of £40.75 at 2.5% per annum?

10. Find the simple interest on £840, 10s. for $4\frac{3}{4}$ y. at 3.75 % per annum.
11. In what time will a sum of money double itself at $2\frac{1}{2}$ % per annum simple interest?
12. What sum of money will produce an interest of £30, 6s. 8d. in $1\frac{1}{2}$ y. at 4 % per annum?
13. A certain sum of money is required to meet a bill of £450 in 16 y. What sum must be invested now at 4 % per annum to realize that sum (simple interest)?
14. Find the present value of £640 obtained in 5 y. at $2\frac{1}{2}$ % per annum.
15. At what rate must £100 be lent so that it will amount to £150 in 12 y.?

Stocks and Shares.—All transactions in stocks and shares are based on the equation $\frac{a}{b} = \frac{c}{d}$. All difficulties of such sums are overcome when a clear idea of the terms used is obtained.

To carry on any business, **Capital** is needed, and this is often raised by the promoters and others interested in the undertaking purchasing **shares**. Thus a capital of £10,000 might be raised in the form of 100 shares at £100 each, or 1000 shares at £10 each or 20,000 shares each of 10s. value, and so on. As the business thrives so the *value* of the shares varies.

A shareholder in a profitable business would find each of his £10 shares worth, say, £15 or £20, i.e. a sum of money greater than its original or *par* value. If he sold his interest in the business he would sell his shares at a *premium* (that is, at a price greater than their face value). On the other hand, if the business has depreciated, a share will be worth less than its face value, and it will be possible to buy a £10 share at, say, £9; a £100 share for, say, £89; and so on. A shareholder would therefore sell at a *discount* (that is, at a price less than the face value of each share). The market value of a share, varying with the prosperity of the business, is termed the **price** of the stock.

Example 1.—A man sells £625 worth of 4 % stock at $93\frac{3}{4}$. What does he realize in cash?

Method—

For £100 stock (i.e. for one share) he gets £ $93\frac{3}{4}$.

For £625 stock he gets x .

$$\therefore \frac{100}{625} = \frac{93.75}{x} \quad x = \frac{625 \times 93.75}{100} = \underline{\underline{£585, 18s. 9d.}}$$

Example 2.—A man spends £1000 in purchase of $4\frac{1}{2}\%$ Railway Debentures at $108\frac{3}{8}$. What amount of stock has he?

£108 $\frac{3}{8}$ will purchase (1 share) £100 stock.

£1000 „ £ x „

$$\therefore \frac{108.375}{1000} = \frac{100}{x}, \quad x = \frac{100 \times 1000}{108.375}$$

= £922.72 worth of stock.

Exercises 32.—Buying and Selling Stock

1. A business is started, the capital being raised by shares each of value £10. What must I give for 25 shares, 16 shares, and 80 such shares at par, and when they are quoted at 9?
2. The $2\frac{1}{2}\%$ Consols are at $82\frac{1}{2}$. What money must I give for £200 worth of this stock?
3. A man has £927 worth of 1907 Argentine stock which he wishes to realize. It stands now at $101\frac{1}{4}$. What money will he receive?
4. What must be paid for 15 of Lipton's ordinary £1 shares, which are quoted at 24s.?
5. Lake View mining shares are nominally worth £1, but are quoted to-day at 16s. 6d. What must be paid for 200 shares?
6. £675 will purchase £750 Great Eastern Railway stock. What is the price of the stock?
7. I gave £1161 for £900 Great Northern Railway 5% stock. What was its quoted price at the time of purchase?
8. How much stock should I obtain by investing £5375 in the Lancashire and Yorkshire Railway stock at $107\frac{1}{2}$?
9. How much is a man at present worth who owns £7350 stock at $87\frac{1}{2}$?
10. I invest £4375.75 in the Bristol Wagon Company stock at 107.5. What stock do I obtain?
11. What is the value of £3505 Great Western Railway stock at $101\frac{1}{2}$?
12. I own £4250 Russian stock. I decide to sell out at 98. What do I get for it?
13. A man invested £330 in the $2\frac{1}{2}\%$ Consols at $82\frac{1}{2}$. How much stock did he buy? When the Consols were quoted at 85 he sold his stock. How much money did he make by the double transaction?

Incomes from Stock.—A man investing in shares receives interest or dividend on his money. Such profits are paid to him yearly or half-yearly, and are reckoned in the usual way of reckoning interest. Care must, however, be taken not to confound “stock” with “cash”. In dealing with every problem relating to interest on or income from stock, first consider if the

money named is already invested, in which case it is "stock"; or if it is to be invested, when it is "cash". The difference is shown in the following examples:—

Example 3.—What income will a man derive from investing £800 in a 4% stock at 75?

Here £800 is "cash" to be invested.

An investment of £75 purchases £100 stock, which brings an income of £4 per annum.

∴ an investment of £800 purchases stock, which brings an income of £ x per annum. I.e.—

$$\frac{75}{800} = \frac{4}{x} \quad 75x = 3200. \quad x = \frac{3200}{75} = \underline{\underline{£42, 13s. 4d.}}$$

Example 4.—What income will a man derive from £800 worth of 4% stock?

Here £800 is "stock", and the price need not be stated since it does not enter into the consideration.

£100 stock pays an annual income of £4.

£800 " " " £ x .

$$\therefore \frac{100}{800} = \frac{4}{x} \quad x = \underline{\underline{£32.}}$$

Exercises 33.—Incomes

1. What income will a man derive from the purchase of £976 Government 2½% Consols?
2. A man has £1000 to invest. He decides to purchase 3½% India stock at 97½. What income will he derive?
3. Chinese 4½% stock is at 101½. What interest will a man annually receive who invests £455, 1s. 3d.?
4. I possess £550 stock paying 2¾%. What interest do I annually derive from this source?
5. A man has £500 to invest. From which will he derive the greater income: (a) 2½% Consols at 83½, or (b) 7% mining stock at 183½?
6. A man has invested £200 in each of the following stocks: (a) India 3½% stock at 96, (b) L.C.C. 3% stock at 88½, (c) Irish Land Loan 2¾% stock at 83¾. What is the total income derived?
7. Which is better: to invest £200 in 5% bonds at 143½ for 2 y., or to lend out the money for that period at 4%?
8. Find the income derived from investment of £100 in 3½% stock at 73½.
9. 3% Buenos Ayres stock is at 68½. What percentage do I get for the money I invest in it?
10. Which is more profitable: 4½% stock at 101½, or 3% stock at 80?

Brokerage.—Business in stocks and shares is usually transacted by brokers. A broker charges for his services a percentage of the value of the business transacted. This is termed “brokerage”. A purchaser has therefore to pay for the stock he buys and to pay the broker as well, i.e. he pays the broker a little more than the price of the stock—usually $\frac{1}{8}\%$, or 2s. 6d. per £100 worth of stock bought. Similarly, on selling stock the vendor receives his money less the broker’s fee of $\frac{1}{8}\%$. Brokerage is easily reckoned on the *price* of the stock. Thus when $2\frac{1}{2}\%$ Consols are at $83\frac{3}{8}$, £100 stock can be bought for $£83\frac{3}{8}$, or $£83\frac{1}{2}$ if the brokerage of $\frac{1}{8}\%$ is included; while, on the other hand, the vendor of £100 stock receives $£83\frac{3}{8} - \frac{1}{8}$, or $£83\frac{1}{4}$ for his stock from his broker. Therefore in buying stock add $\frac{1}{8}\%$ to the price, and in selling subtract $\frac{1}{8}\%$ from the price, in order to obtain the value of the stock to the public.

Example 5.—What income will a man derive from investing £750 in the $3\frac{1}{2}\%$ stock at $67\frac{3}{8}$, reckoning brokerage.

By investing £67 $\frac{3}{8}$ and paying £ $\frac{1}{8}$ brokerage a man has stock yielding £3 $\frac{1}{2}$ interest. I.e.—

£67½ purchases an income of £3½.

∴ £750 „ „ £x.

$$\text{I.e. } \frac{67.5}{750} = \frac{3.5}{x}. \quad x = \frac{3.5 \times 750}{67.5} = \underline{\underline{\text{£}38, 17s. 9\frac{1}{2}d.}}$$

Exercises 34.—Stocks (Brokerage of $\frac{1}{8}\%$)

1. How much must be paid to the broker that he may purchase for you one share of each of the following:—(a) Consols at $93\frac{3}{4}$, (b) Local Loans at $95\frac{1}{4}$, (c) Great Western Ordinary at $120\frac{1}{8}$, (d) Midland Deferred at $57\frac{7}{8}$, (e) L. and N. Western Ordinary at $133\frac{3}{8}$, (f) Chinese at par?
2. How much is realized per share by sale of 1 share in each of stocks mentioned in Exercise 1?
3. I sell out £476 stock from 3% stock at $87\frac{3}{8}$, and purchase with the proceeds 4% stock at $94\frac{1}{8}$. What is the change in my income?
4. What will be the cost of £750 Midland Railway stock at $137\frac{3}{8}$?
5. How much can be realized by the sale of £1350 Railway stock at $101\frac{1}{2}$?

6. Find the annual income derived from the purchase of £500 worth of $4\frac{1}{2}\%$ stock at $97\frac{1}{8}$. What money must be paid down to obtain this stock?
7. A $2\frac{1}{2}\%$ stock is quoted at $82\frac{1}{8}$. What must be paid to purchase £660, 10s. stock?
8. A man buys Chinese $4\frac{1}{2}\%$ stock at $96\frac{1}{8}$. What percentage interest does he get on his cash?
9. What must be paid to purchase £350 stock at $95\frac{3}{4}$?
10. What is realized by selling £530 stock quoted at $120\frac{1}{4}$?
11. By investing a certain sum of money in the $3\frac{1}{2}\%$ stock at $140\frac{5}{8}$ I get an income of £150 per annum. What sum did I invest?

Graphs.—The equality of two ratios—or proportion—has been so far demonstrated in two ways:—

(a) Algebraically by the equation $\frac{a}{b} = \frac{c}{d}$;

(b) geometrically by using the sides of similar triangles.

An extension of (b) gives the **Graph**, or graphical representation of the relations existing between items of two series of quantities.

The principle and the utility of the graph will be seen by a careful consideration of the following example.

If a railway company can issue an excursion ticket for a distance of 20 m. for 1s. 3d., what should be the cost of a similar ticket for distances of 50 m., 30 m., and 80 m.?

By the equation $\frac{a}{b} = \frac{c}{d}$ we can laboriously arrange the results thus—

$$\left. \begin{aligned} \frac{20}{50} &= \frac{15d.}{x} & x &= 37.5d., \text{ or } 3s. 1\frac{1}{2}d. \\ \frac{20}{30} &= \frac{15d.}{x} & x &= 22.5d., \text{ or } 1s. 10\frac{1}{2}d. \\ \frac{20}{80} &= \frac{15d.}{x} & x &= 60d., \text{ or } 5s. \end{aligned} \right\}$$

The results may then be arranged thus—

Distances in miles.....	20	30	50	80
Cost of excursion tickets	1s. 3d.	1s. 10 $\frac{1}{2}$ d.	3s. 1 $\frac{1}{2}$ d.	5s.

The graph (fig. 18) gives these results immediately, and as many more as are needed, and on inspection only.

HOW IS THIS GRAPH OBTAINED?—Set off OX and OY at right angles to one another. These are called **axes**, OX being spoken of as the “axis of x ” and OY as the “axis of y ”. On these mark off distances to represent distances in miles and cost of tickets, just as was done in making similar triangles represent

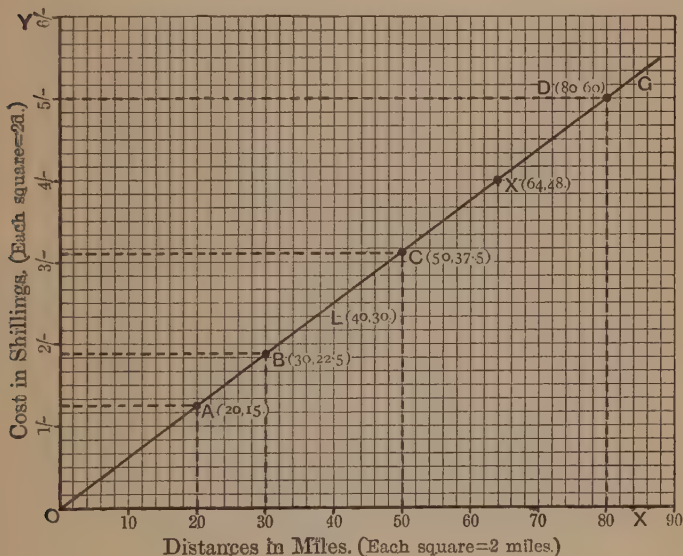


Fig. 18

proportion (*vide* fig. 17). The squared paper permits this to be done easily and carefully. In fig. 18 every square on the axis of x has been made to represent 2 m., and every square on the axis of y has been made to represent 2 pence. A dotted line drawn from the point on the axis of x , representing 20 m., meets the dotted line drawn from a point on the axis of y , representing 1s. 3d. at the point A. Thus the point A represents by its position both 20 m. and 1s. 3d., and for this reason may be spoken of as the point 20, 15 (i.e. 20 m. 15 pence). Join A to the origin O and produce it. This gives the graph OG , and OG

passes through the points B, C, and O, which represent respectively 30 m., 1s. 10½*d.*; 50 m., 3s. 1½*d.*; and 80 m., 5s. To find the required answers to the sum it is only necessary to follow the squared-paper lines from the required distances on the axis of *x* vertically to the graph OG, and then to proceed in a horizontal direction from these points to the axis of *y*, where the required prices can be read off. The line from D to the axis of *y* gives 5s. quite easily, but the similar line from C falls between the distances representing 3s. and 3s. 2*d.* Care has therefore to be taken both in drawing the graph and in reading off values, though there should be no difficulty in reading off the prices corresponding to a quarter of a square (i.e. ¼ of 2*d.*, or ½*d.*). The dotted line from C to the axis of *y* obviously lies nearer to the 3s. 2*d.* than the 3s. line. Similarly, the dotted line from B to the axis of *y* lies one-quarter of the distance between the 1s. 10*d.* and 2s. lines. The price for 30 miles is therefore 1s. 10½*d.*

The graph OG gives more than the prices of tickets for the stated distances. It forms, in fact, a price list from which a railway clerk could read any number of distances with their excursion fares.

For example, he is asked for an excursion ticket to a place 64 miles distant. X on the graph is the point (64, 48). He charges therefore 4s. It is also seen from L on the graph that a ticket costing me half a crown will entitle me to travel 40 m.

The principle underlying the graph is simple. Here are two series of similar triangles. The distances on the axis of *x* are proportional to the corresponding distances on the graph OG, and these in turn are proportional to distances on the axis of *y*. For example—

$$\frac{20 \text{ miles}}{30 \text{ miles}} = \frac{OA}{OB}; \text{ also } \frac{OA}{OB} = \frac{1s. 3d.}{1s. 10\frac{1}{2}d.}$$

$$\therefore \frac{20 \text{ miles}}{30 \text{ miles}} = \frac{1s. 3d.}{1s. 10\frac{1}{2}d.}$$

The graph is therefore only an extension of the “similar triangles” method of representing proportion.

Exercises 35

1. Taking each square on the axes of x and y to represent one, show the position of the following points:—(6, 8), (14, 16), (4, 12), (13, 6).
2. Represent on squared paper the points:—A (6, 8), B (14, 16), C (0, 8), D (8, 16). Join these points, and name the figure ABDC.
3. A (7, 10) and B (16, 20) are joined. What is the position of the central point of AB?
4. Work by graph the following:—If 5 gr. of silver are worth a penny, what should be the weight of silver in a shilling, florin, half-crown, and crown?
5. Given that 1 m. is 40 in., find by graph how many inches are equivalent to 12 cm., 8 cm., 85 cm., and how many cm. there are in 20 in., 15 in., 12 in., and 8 in.?
6. 25 fr. equal one English sovereign. Draw a graph to express any sum of English money to £1 as French money in francs and centimes. What is the value in shillings of 12 fr. 50, 18 fr. 75, and 10 fr., and how many francs can I get for 8s., 7s. 6d., 16s., and 15s.?
7. A kilogram weight is 2.2 lb. Find by graph the corresponding weights in lb. for 500 g., 300 g., and 800 g.
8. A dividend is declared of 4s. in the £1. Make a graph to show all dividends payable on sums to £10, and state the amount paid on £3, 10s., £6, 15s., and £7, 10s.
9. A bankrupt can pay 12s. in the £1. Show by graph the amounts realized on debts of £10, £7, 10s., and £6, 15s. What was the amount of the debt if the creditor in this case received 34s.?
10. A bookseller gives 25 % discount on the list price of books in his catalogue (i.e. a book listed at 10s. can be sold at 7s. 6d.). Make a graph showing list prices and selling prices, in which the latter are 25 % below the former. What must I pay for a book marked 6s., and what value book should I receive for 3s. 6d.?

The chief difficulty in making a graph lies in the determination of appropriate scales for the axes. In deciding on the scale regard must be had to—

1. The smallest amounts which have to be read.
2. The limits of the paper, care being taken that the largest amounts stated shall appear on the scale.

Thus in fig. 18, by making each square on axis of y represent one shilling, sixpence, or even threepence, it would have been difficult to estimate the odd halfpence, which was readily done by making each square represent twopence. On the other hand,

a far better plan would have been to make each square represent a penny or a halfpenny, but in that case prices to 6s. could not have been shown, as the paper was not sufficiently large. It is necessary, therefore, to adapt the scales to these two considerations, remembering at all times to make the graph as large as the given data and squared paper permit.

Graphs do not always go through the origin 0, nor are they always straight lines. In the case of the graph representing equal ratios the graph is a straight line, but when the relationships existing among other varying quantities are graphed, a curve often results.

Example 1.—The relationship existing between the height of a barometer and the altitude of the place where the readings are made is shown by fig. 19.

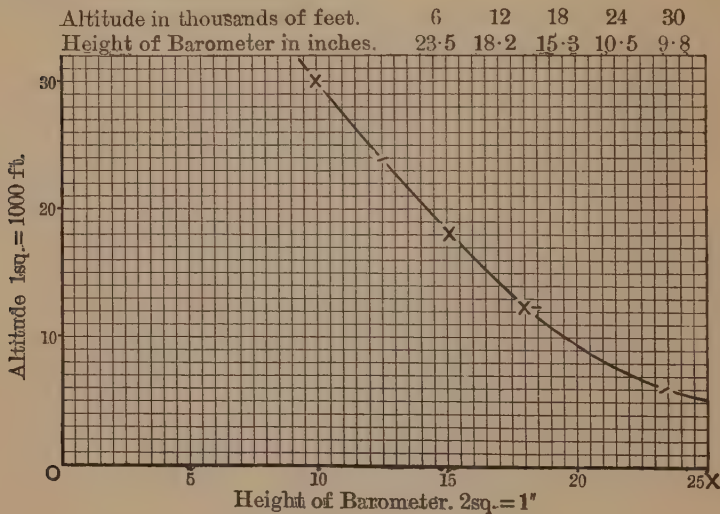


Fig. 19

Example 2.—Density of water given at different temperatures:—

Temp.....	0	1°	2°	4°	7°	12°	15°	17°
Density...	·99988	·99993	·99997	1·0	·99994	·99955	·99915	·99884

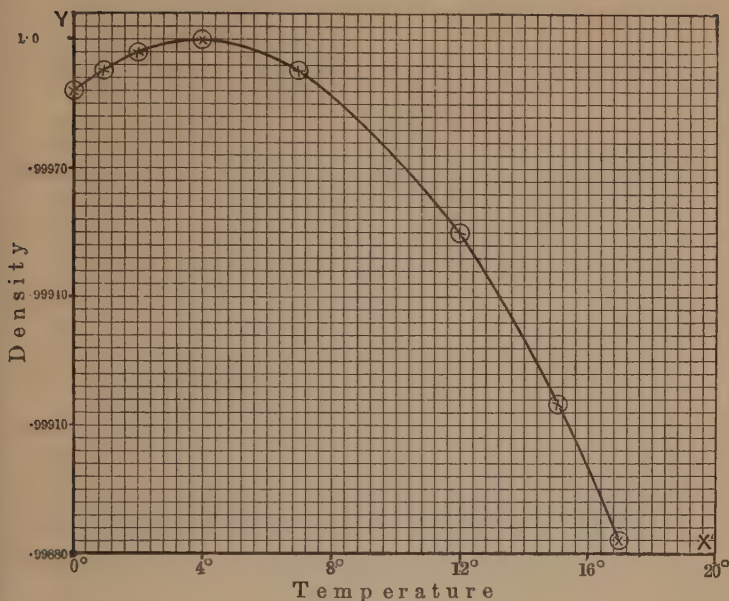


Fig. 20

Note.—The marking off of the scale on the “Density” axis gives some indication of how to proceed in difficult cases. In this case the largest amount to be shown was 1.0, and the smallest was .99884. Space limited this difference of $(1 - .99884 = .00116)$ to some 40 squares. Now .00116 is approximately .00120, and $120 \div 40 = 3$. Thus by taking 1 square to represent .00003, or 10 squares to represent .0003, and by commencing the scale with .99880, the squared paper is found to accommodate the required data.

Exercises 36

- The following data give the approx. population in millions in England and Wales in the given years. Represent this by a graph.

Years ...	1831	1841	1851	1861	1871	1881	1891
Population in millions }	14	16	18	20	23	26	29

2. Plot x and y on squared paper.

x ...	5	16	26	35	50
y ...	13	22	24	27	31

3. The pressures exerted by saturated water vapour at different temperatures is given. Plot these data, reducing them to graph form.

Temperature	17°	26°	30°	37°	43°	51°	61°	66°	76°
Pressure ...	1.1	1.5	2.0	3.6	5.1	9.0	14.9	18.3	29.8

4. Show by graph the relationship existing between the volume of a gas in a cylinder and the pressure to which it is subjected if the following data are obtained from observations:—

Volumes in C in	49.9	43.5	40.8	36.8	31.0	24.8
Pressures in lb. } per sq. in.	75.5	86.5	92.3	102.2	121.0	151.9

5. At the following draughts in sea water a particular vessel has the following displacements:—

Draught in feet ...	15	12	9	6.3
Displacement in tons	2098	1512	1018	586

What are the probable displacements when the draughts are 11 and 13 feet respectively? (S. and A. 1901.)

6. A is the horizontal sectional area of a vessel in square feet at the water level h , being the vertical draught in feet.

A	14850	14400	13780	13150
h	23.6	20.35	17.1	14.6

Plot on squared paper and read off and tabulate A for values of h , 23, 20, 16. (S. and A. 1902.)

7. The following are the areas of cross section of a log at right angles to its straight axis:—

Area in sq. in.	250	292	310	273	215	180	135	120
x in. from end	0	22	41	70	84	102	130	145

Plot A and x on squared paper. What is the probable cross section at $x = 50$ in.? (S. and A. 1903.)

3. Make a price list for glass beakers of given capacities in cubic centimetres quoted at x / per doz.

Capacity in C.C.	100	300	500	800	1000
Price in shillings	2/3	4/	6/	10/	12/6

What should be the price of a 750 c.c. beaker, and one holding 250 c.c.?

INVERSE PROPORTION

I.e. $\frac{a}{b} = \frac{d}{c}$ instead of $\frac{a}{b} = \frac{c}{d}$ (See Exercise 26 and

Comparison of Ratios.)

Consider the following examples of proportionals, and it will be immediately seen that one ratio has to be inverted to give equality.

Example 1.—A man does a piece of work in, say, 24 days.

Two men do this piece of work in, say, 12 days, working at the same rate.

Three men do this piece of work in, say, 8 days, working at the same rate.

Thus, with the *increase* of the number of men, there is a corresponding *decrease* in the time taken to do the work.

If 12 men do a piece of work in 6 days, how long will it take 8 men?

Ratio of men, $\frac{12}{8}$. Ratio of days, $\frac{6}{x}$.

But since a *smaller* number of men will take *longer* to do the work, i.e. the inverse of the usual order of things.

$\therefore \frac{12}{8} = \frac{x}{6}$ (I.e. $\frac{6}{x}$ is inverted to meet the special case.)

$$8x = 72.$$

$$x = 9 \text{ days.}$$

Example 2.—Construct a square of 3" side and rectangles 2" $\times$ 4.5", 4" $\times$ 2.5", 5" $\times$ 1.8", 6" $\times$ 1.5". Set these side by

side and observe. You have *equal areas*, but the *lengths* and *breadths* vary. Further, to maintain equal areas the *lengths increase* as the *breadths proportionally decrease*, and vice versa. Thus the lengths vary *inversely* as the breadths. The same truth obtains in the case of triangles. To construct triangles of equal area it is necessary to shorten the bases as the altitudes are increased, and vice versa. Try this on squared paper, commencing with a triangle of 20 squares base and 24 squares altitude.

Example 3.—Balance a metre ruler on some wedge-shaped article as a fulcrum—the finger is hardly reliable in cases of careful working. You find it balances at the 50-cm. mark if it is uniform. Place a 20-g. weight on the 30-cm. mark (i.e. 20 cm. from the fulcrum). To restore equilibrium with a 10-g. weight you will find it necessary to place it on the 90-cm. mark (i.e. 40 cm. from the fulcrum). Similarly, equilibrium restored with a 40 g. weight, finds that weight placed 10 cm. from the fulcrum.

Again, 10 g. placed 25 cm. from the fulcrum balances 5 g. placed 50 cm. away on the other side, or 25 g. placed 10 cm. on the other side. Try these experiments.

In each case you have a balance, and summed up your results are:—

A Weight used.	Distance from Fulcrum.	B Weight used.	Distance from Fulcrum.
20 g.	20 cm.	10 g.	40 cm.
20 g.	20 cm.	40 g.	10 cm.
10 g.	25 cm.	5 g.	50 cm.
10 g.	25 cm.	25 g.	10 cm.

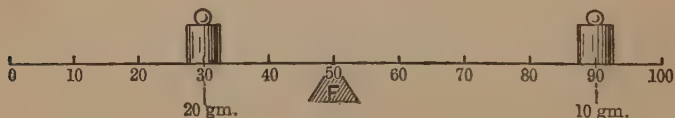


Fig. 21

Both from the summary and from fig. 21, but best of all from your actual experiments in this direction, it will be obvious that to have a *balance* with *unequal weights* their distances from the fulcrum must vary. The *greater weight* is always *nearer* the fulcrum, and vice versa. In fig. 21 the ratio of the weights

is $\frac{20}{10}$, i.e. $\frac{2}{1}$, but the ratio of their distances from the fulcrum is $\frac{20}{40}$, i.e. $\frac{1}{2}$.

Thus the ratio of the weights is the *inverse* of the ratio of their distances from the fulcrum.

In the above three examples the items of one set of variables (time, lengths, weights) do not *vary directly* as those of the other set (men, breadths, distances). With the increase in one set there is a corresponding decrease in the other. The items of one set are therefore said to *vary inversely* as those of the other.

To express "Direct" and "Inverse" Variation.—Because the area of a rectangle "varies directly as", i.e. increases or decreases with, the length and breadth of its sides, the expressions $A \propto l$ and $A \propto b$ were used to convey this idea. Similarly, $C \propto D$ conveys the idea that the circumference of a circle increases and decreases with (i.e. "varies directly as") its diameter.

$A \propto l$, $A \propto b$, $C \propto D$, $x \propto y$ express "direct variation".

Since, in Example 1 above, the time required "varies inversely as" the men employed, i.e. increases if the number of men be decreased, and vice versa, the expression $T \propto \frac{1}{M}$ or $M \propto \frac{1}{T}$ is used to convey this idea.

Note.—This example can be viewed differently. If 1 man takes 12 d., then 1 man does $\frac{1}{12}$ of the work in 1 d. Similarly, 2 men would take 6 d., or each man doing $\frac{1}{12}$ of the work per day, 2 men doing $\frac{1}{6}$ of the work per day.

Ratio of men $\frac{1}{2}$, ratio of times $\frac{12}{6}$ or $\frac{2}{1}$ (inverse ratio); ratio of amount of work done $\frac{\frac{1}{12}}{\frac{1}{6}} = \frac{6}{12} = \frac{1}{2}$ (direct ratio).

Thus "inverse" ratio sums involving men and time become "direct" when amount of work per day is substituted for time.

Similarly, in Example 3 above, $W \propto \frac{1}{D}$ expresses the idea that the weights increase as their distances from the fulcrum decrease, i.e. that the weights employed vary inversely as their distances from the fulcrum.

$$T \propto \frac{1}{M}, \quad W \propto \frac{1}{D}, \quad x \propto \frac{1}{y}$$

express "inverse variation".

Note.— $\frac{1}{M}$ is the reciprocal of M , just as $\frac{1}{8}$ is the reciprocal or the “inversion” of 8 or $\frac{8}{1}$.

Exercises 37.—Work Mentally

1. If 10 men do a piece of work in 5 d., how long will it take 20 men? Express the ratios of the men, the days taken, and the amount of work done per day.
2. A triangle has a base 4 in. and an altitude of 3 in. To keep the area constant, what height must I make it if I reduce its base to 2 in.?
3. A uniform 12-in. ruler balances about its 6-in. mark. Where must a 2-oz. weight be placed to balance a 1-oz. weight placed on the 10-in. mark?
4. The volume (V) of a gas varies inversely as the pressure (P) to which it is subjected, i.e. the volume *increases* as the pressure is *decreased*, and vice versa. Express the idea mathematically. When the pressure is 14 lb. to the square inch the volume is 10 c. in. What would you expect the volume to be when the pressure was (a) doubled, (b) halved?
5. The strength of an electric current varies inversely as the resistance of the circuit. Does $C \propto R$ or $C \propto \frac{1}{R}$ best express this idea? If a certain electric force produced a current of 2 amp. in a circuit whose resistance was 5 ohms, what would be the strength of the current if the resistance were (1) increased to 10 ohms, (2) reduced to 1 ohm?
6. A train travelling 30 m. per hour accomplishes its journey in one hour and a half. How long would a train travelling 45 m. per hour take to make the journey?
7. 10 cwt. can be carried by train to a certain town for 2s. 6d. What distance should 2 cwt. be carried for the same money?
8. x varies inversely as y . When x is 2, y is 10. What should x be when y is 5?

To work Examples involving Inverse Ratios.

1. If 12 men do a piece of work in 15 d., how long will it take 20 men?

$$\begin{array}{l} 12 \text{ men require } 15 \text{ d.,} \\ 25 \text{ „ „ „ } x \text{ „ } \end{array} \left. \vphantom{\begin{array}{l} 12 \text{ men require } 15 \text{ d.,} \\ 25 \text{ „ „ „ } x \text{ „ } \end{array}} \right\} \text{but } M \propto \frac{1}{D}. \\ \therefore \frac{12}{25} = \frac{x}{15}. \quad 25x = 12 \times 15. \quad x = 7\frac{1}{5} \text{ d.}$$

2. A metre ruler balanced about its 50 cm. mark has a weight of 12 g. placed 40 cm. from the fulcrum, and an object of un-

known weight placed 25 cm. from the fulcrum. Find the weight of the object.

Method 1.—12 g. weight placed 40 cm. from the fulcrum balances x g. weight placed 25 cm. from the fulcrum.

$\therefore \frac{12}{x} = \frac{25}{40}$, since weights vary inversely as distances, or

$$W \propto \frac{1}{D}.$$

$$x = \frac{12 \times 40}{25} = \frac{96}{5} = 19\frac{1}{5} \text{ g.}$$

Method 2.—By equality of “moments”.

The product of a weight and its perpendicular distance from the fulcrum is termed the “moment” of the weight about this point.

The cross multiplication of $\frac{12}{x} = \frac{25}{40}$ obtained in Method 1 gives $25x = 12 \times 40$, i.e. the weight x multiplied by its distance 25 cm. from the fulcrum (moment of x) equals the weight 12 g. multiplied by its distance (40 cm.) from the fulcrum (moment of 12 g.).

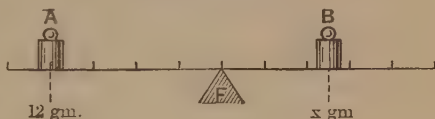


Fig. 22

Thus, calling the weights A and B, the sum can be worked thus—

$$\text{Moment of A} = \text{Moment of B.}$$

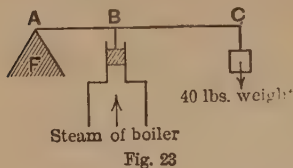
$$\therefore 12 \times 40 = 25x.$$

$$x = \frac{12 \times 40}{25} = 19\frac{1}{5} \text{ g.}$$

3. The arms of the lever of a safety valve are 2 in. and 8 in., and at the end of a longer arm a weight of 50 lb. is hung. What must be the pressure of steam in the boiler to just raise the valve?

The arm AC is hinged at A, and, resting on B, carries a weight of 40 lb. at C.

When the pressure of steam in the boiler is sufficient it forces up the piston in the safety valve at B. Thus the duty of the 40 lb. weight is to keep the valve closed and the steam from escaping until the required pressure has been attained in the boiler. Then the automatic opening of the valve by the steam keeps the pressure in the boiler within the region of safety.



In this case $AB = 2$ in., $AC = 8$ in.

Method 1.—Let x be pressure of steam.

Then x lb. at 2 in. from fulcrum balances

40 „ 8 „ „

It is obvious that the lengths of the arms are inversely as the two forces (viz. that of the steam and weight). I.e.—

$$\frac{x}{40} = \frac{8}{2}, \text{ or } x = \frac{40 \times 8}{2} = 160 \text{ lb.}$$

Method 2.—

Moment of x = Moment of 40 lb. weight.

$$2x = 40 \times 8.$$

$$x = 160 \text{ lb.}$$

N.B.—If the safety valve be 1 sq. in. in cross section then the pressure of the steam would be 160 lb. per square inch; if $\frac{1}{2}$ sq. in. in cross section the pressure = 320 lb. per square inch; and so on.

Exercises 38

1. If 75 men finish a piece of work in 24 d., how many men will be required to finish it in 40 d.?
2. If 12 men dig a foundation in 8 d., in what time will 30 men do the work, working at the same rate?
3. A vessel sailing 16 knots per hour takes 50 d. to make her voyage. How long will it take the vessel, steaming under similar conditions, at the rate of 20 knots per hour, to make the voyage?
4. If 15 cwt. 2 qr. of goods can be carried 60 m. for 8s., how far ought 3 cwt. 1 qr. to be carried at the same rate?
5. A pipe discharging water at the rate of 3 qt. per minute fills $\frac{1}{2}$ of a tank in 6 min. How long would it take to fill the tank if the pipe discharged at the rate of a gallon a minute?

6. The areas of two triangles are equal. The base of one is 15.6 in., and its perpendicular height 8.37 in. If the base of the other is 26.4 in., what is its height?
7. Two triangles equal in area have their altitudes respectively 17.25 cm. and 23.8 cm. If the base of the first is 1 dm., what is the length of the base of the other?
8. A rectangle measures 16.25 in. by 8.75 in. Find the breadth of a parallelogram equal to the rectangle in area, whose length is 12.3 in.
9. A straight lever AB is 4 ft. long, and is balanced about its middle point. Find what weights must be placed 6 in. from A to balance (a) 5 lb., (b) 6 lb., (c) 10 lb. placed at B.
10. A uniform yardstick is balanced about its middle point, and a pound weight is placed $14\frac{1}{2}$ in. from the fulcrum to balance an unknown weight at 10 in. from the fulcrum. What is the weight at 10 in. distance?
11. If in Exercise 10 the unknown weight had been placed a foot from the fulcrum, where should the 1-lb. weight have been placed?
12. Find the weight of a 12-in. ruler if when a weight of 1 oz. be suspended from one end it balances about a point 2 in. from that end. (*Note.*—The weight of the ruler acts through its middle point.)
13. An electric current varies inversely as the resistance of the circuit through which it flows. When the resistance is 15.8 ohms the strength of the current is 14.6 amp. What would be the current strength if the resistance to be overcome was 10.7 ohms?
14. The volume of a gas varies inversely as the pressure to which it is subjected. The volume of a gas measures 16.3 c.c. What will its volume be if the pressure to which it is subjected is (a) increased 5 times, (b) increased 1.5 times, (c) halved?
15. The volume of a gas subjected to 2 atmospheres pressure is 15.7 c. in. What will be its volume when it is subjected to a pressure of 4.75 atmospheres?
16. The arms of the lever of a safety valve are 2 in. and 1 ft. 6 in., and at the end of the longer arm a weight of 20 lb. is attached. What is the pressure of steam on the valve to just open it? If the valve is $\frac{1}{2}$ sq. in. in area, what is the pressure of steam in the boiler per square inch?
17. Find the pressure of steam in a boiler just sufficient to raise a safety valve of 1 sq. in. area when the arms of the lever are 1 in. and 18 in. and a 12-lb. weight is attached to the longer arm.
18. The arms of the lever of a safety valve are $1\frac{1}{2}$ in. and 20 in., and at the end of the longer arm a weight of 20 lb. is hung. If the area of the valve be $1\frac{1}{2}$ sq. in., what is the total pressure of steam per square inch allowed in the boiler?

THIRD, FOURTH, AND MEAN PROPORTIONALS

$\frac{a}{b} = \frac{c}{d}$ represents the equality of two ratios (i.e. a proportion sum). The terms a , b , c , and d are called the **Proportionals**.

When $\frac{a}{b} = \frac{c}{d}$ the proportionals are a , b , c , d , of which d is termed the **fourth proportional**. When $\frac{a}{c} = \frac{c}{d}$ the proportionals are a , c , c , d . d is here the **third proportional**, since c occurs twice; further, c is termed the **mean proportional** of a and d , since a bears the same relationship to it that it does to d , or since $\frac{a}{c} = \frac{c}{d}$.

Example 1.—Find the fourth proportional of 6, 8, and 12.

$$\frac{6}{8} = \frac{12}{x}. \quad 6x = 8 \times 12. \quad x = 16.$$

16 is the fourth proportional.

Example 2.—Find the third proportional of 6 and 10.

$$\text{i.e. } \frac{6}{10} = \frac{10}{x}, \text{ or } 6x = 100. \quad x = 16\dot{6}.$$

16.6 is the third proportional.

Example 3.—Find the mean proportional of 3 and 27.

$$\frac{3}{x} = \frac{x}{27}. \quad x^2 = 3 \times 27 = 81. \quad \therefore x = 9.$$

9 is the mean proportional.

Note.—The graphic representation of a mean proportional is found in a future section. Similar triangles furnish a graphic representation of third and fourth proportionals.

Exercises 39

1. Find the fourth proportionals of:—

(a) 6.25, 7.3, 8.9.

(b) 7 in., 9 in., £.75.

(c) 4.362, 9.758, 6.384. (Use contracted methods.)

(d) 4.362×10^2 , 1.823×10^3 , 3.57.

2. Find the third proportionals of: (a) 6 and 16, (b) 8.372 and 6.47, (c) 5.432 and 7.893, (d) 16 ft. and 12 yd.
3. Find the mean proportionals of: (a) 5 and 125, (b) 14 and 36, (c) 12.6 and 19.68, (d) 2 in. and 18 in.

SUMMARY

Magnitudes of one kind depend upon magnitudes of another kind. For example: The money realized by the sale of a certain kind of article increases with the number sold; the intrinsic value of an article is great or small according to the material of which it is composed and the labour expended upon it; the time occupied in performing any work bears some relationship to the number of men employed to do it; the volume of a gas depends upon the pressure to which it is subjected. Every quantity influences and is influenced by other quantities, and it is the purpose of Section III to endeavour to fix this existing relationship on a mathematical basis.

A **Ratio** is the relationship existing between magnitudes of the same kind. It tells how many times one is greater than another, and the rate at which prices and other dependent considerations should vary. The ratio between two amounts a and b is expressed thus: $\frac{a}{b}$.

Interest, Stocks, Price Lists, Percentages, Levers, &c., present at least two series of magnitudes which influence one another in one of two ways. If the increase of magnitudes of one series is followed naturally by a corresponding increase in the magnitudes of the other series, and vice versa, the magnitudes are said to be "directly proportional", but when an increase in one series means a corresponding decrease in the other series, the magnitudes are said to be "inversely proportional".

Examples of **Direct Proportion** are found in dealing with Price Lists, Percentage, Stocks, Interest, &c. For example, an interest of 4 % per annum infers that a sum of £100 lent for 1 year earns £4 interest; a larger sum than £100 realizes a correspondingly larger interest; a smaller sum, a smaller interest. Thus the amount of interest earned varies directly with the amount of money lent, or $I \propto P$ (where I = Interest and P

= Principal). The interest earned by any principal lent also "varies directly" as the time during which it is allowed to earn the increment, or $I \propto T$ (where T = Time of loan).

In the case of Direct Proportion, if $\frac{a}{b}$ is the ratio of the magnitudes of one series and $\frac{c}{d}$ the ratio of the corresponding magnitudes of the other, then $\frac{a}{b} = \frac{c}{d}$ is the general form of the sums bearing on these considerations.

Examples of **Inverse Proportion** are found in dealing with Time sums, Levers, Hydrostatic and other laws, Safety Valves, &c. For example, the strength of electric current flowing in a circuit decreases as the resistance increases, and vice versa. Similarly, the rate at which water flows along pipes depends on their size—the greater the friction owing to small-size pipes and the slower the flow. Thus if C represents current and R represents resistance, the current varies inversely as the resistance, or $C \propto \frac{1}{R}$. In the case of Inverse Proportion, if $\frac{a}{b}$ is the ratio of the magnitudes of one series, and $\frac{c}{d}$ is the ratio of the two magnitudes of the other series, then $\frac{a}{b} = \frac{d}{c}$ is the general form to which all sums bearing on these considerations can be reduced.

The sides of **Similar Triangles** similarly placed are proportional. Similar triangles can be used to graphically illustrate proportion sums, by representing the proportional quantities to suitable scales upon the similarly placed sides. *Graphs* are lines upon squared paper showing the relationship existing between two series of magnitudes. Care must be taken in marking off the scales upon the axes, that the smallest necessary reading may be taken with sufficient accuracy, and yet permit all the given data to be utilized in its construction.

Simple Interest.—The chief terms used are *Principal* or money lent, *Rate per cent*, or the profit gained on every hundred units of principal lent, the *Time* during which the money earns *Interest*, and the *Sum* or *Amount* repayable to the lender, and which is made up of the sum of principal and interest. These "varying" quantities are connected by the formula $100 \times I = P \times R \times T$.

Stocks and Shares involve the use of a larger number of terms. *Capital* is raised by means of *shares* or part ownerships in the business concern. These *shares* entitle their holders to a participation in the profits or *dividends* of the business. Shares can be sold and bought like every other commodity, the agent being a *broker*, who charges *brokerage* for his labour. This brokerage usually amounts to one-eighth per cent of the business done. The *value* or *price* of the stock or share depends on the state of business. In an exceptionally flourishing concern, stocks and shares are sold at a *premium*, or above their *face* value. If they realize their face value they are said to be at *par*, if less they are at a *discount*. Every problem in stocks depends on a clear comprehension of these terms, and on an accurate knowledge of proportion.

Levers.—The distances of two masses from the fulcrum along the arms of a balance when the system is in equilibrium are inversely proportional to the magnitude of their weights. If A and B are the weights and *c* and *d* are their distances from the fulcrum when the whole system is in equilibrium, then

$$\frac{A}{B} = \frac{d}{c}, \text{ or } Ac = Bd.$$

Ac and *Bd* are each the product of the weight and its perpendicular distance from the fulcrum. This product is termed the *Moment* of the Force (weight) about the fulcrum.

SECTION IV.—PLANE ANGLES.

A **plane angle** is the inclination of two lines in a plane to one another which meet at a point. Section IV will only deal with plane angles or angles made in one plane. Some of the plane figures are named after their angles, e.g. triangle, quadrangle, rectangle.

THE MEASURE OF AN ANGLE

Various methods of measuring this inclination of two lines to one another have been adopted and are in use, and since each

method has certain peculiar practical advantages, there has been no effort to reduce these methods to one general mode of reckoning.

An angle is measured—

1. By comparing it with a standard angle called a **right angle**.

2. By dividing the quadrant of a circle into a definite number of parts, and then expressing the given angle as so many of these parts—i.e. as **degrees** or **grades**.

3. By comparing the *arc* subtending the angle with its *radius*—i.e. as **radians**.

4. By taking the measure of the **chord** of this arc.

5. By making the angle one angle of a right-angled triangle, and then by instituting a comparison of the sides, expressing its measure as a ratio—i.e. as **sine**, **cosine**, or **tangent**.

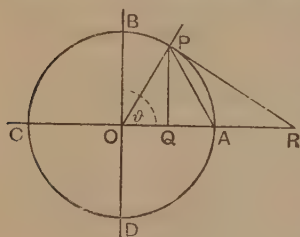


Fig. 24

Each of these methods, its relation to and conversion into other methods of measuring angles, and its particular practical applications, will be considered in turn, but it is well to have a clear idea of the principles underlying these methods first, in order to properly follow the work as it is set out.

Fig. 24 gives the key to all five modes of measurement set out above. It should be carefully studied, and the questions on it attempted before proceeding.

ABCD is a circle divided at O into 4 quadrants or 4 right angles. AOP is the angle (it is marked θ) which is to be measured; if angle θ becomes smaller, then P will fall nearer to A on the circumference of the circle; while if angle θ increases, P will be farther from A. The straight line PA is the *chord* subtending angle θ . PQ is a *perpendicular* meeting OA in Q, and OQ is the *base* of the triangle OQP. PR is the *tangent* to the circle from the point P, meeting OA in R.

Questions on Methods of Measuring Angles

1. As angle θ increases or decreases does the length of the arm OP vary? Give your reasons.
2. As angle θ increases in size what must happen to the length of the arc AP, and to the length of the chord PA? Will the opposite result with a decrease in the size of angle θ ?
3. How does the perpendicular PQ behave as the angle θ decreases or increases?
4. What happens to OQ when θ is made a smaller angle?
5. PR is always at right angles to OP, since it is a tangent. Now if P moves nearer to A (that is angle θ gets smaller) what happens to point R, and to the length of the tangent?
6. If the angle θ were doubled, would the resulting angle be *obtuse* (i.e. larger than a right angle) or *acute* (i.e. smaller than a right angle)?
7. If the arc AP were halved and the mid point joined with O, what relationship will each of the smaller angles so formed bear to the angle θ ?
8. Where will point P come in regard to A if the perpendicular PQ is halved? What effect has this upon angle θ ?
9. I desire to shorten the base OQ, what must be done to angle θ to still keep P on the circumference of the circle?

You will, from the above questions, note that the perpendicular PQ, the chord PA, the arc PA, and the tangent PR *vary directly* in length as the angle θ , but that the base OQ *varies inversely* as the angle θ , while OP remains of constant length. By noting the dependence of these lengths on the size of the angle, you have gained the clues to angle measurement, for—

1. The variation in the length of the arc with the size of the angle has given rise to a measurement of angles by arcs, i.e. method of radians.
2. The variation of the chord with the angle permits *chords* to be taken as a measurement of angles.
3. The variation (direct) of the perpendicular and the variation (inverse) of the base of the triangle POQ with the size of the angle θ has given rise to a consideration of *sines* and *cosines*.
4. The dependence of the length of the tangent on the size of the angle has developed a consideration of *tangents* as the measures of angles.

MEASUREMENT IN DEGREES AND GRADES

The construction of an angle and its measurement in degrees have been simply considered in Section I. The most common mode of measuring an angle is that in which a *right angle* is divided into 90 *degrees*, each degree (1°) being subdivided into 60 *minutes* of arc ($60'$), and each minute of arc being further subdivided into 60 *seconds* of arc ($60''$). $15^\circ 12' 36''$ means 15 degrees 12 minutes 36 seconds.

The French decimal or metric division of a right angle is that in which 100 *grades* equal 1 right angle, 100 *minutes* of arc equal 1 grade and 100 *seconds* equal 1 minute. $15^g 14' 36''$ means 15 grades 14 minutes 36 seconds.

Thus $90^\circ = 100^g$, and the conversion of degrees to grades, and vice versa, depends upon this. So the measure of an angle in degrees must bear the same relationship to 90° as its measure in grades does to 100^g . Otherwise its measure in degrees is just the same fraction of 90 as its measure in grades is of 100. So—

$$\frac{x^\circ}{90^\circ} = \frac{y^g}{100^g}. \quad \left(\text{Cf. } \frac{a}{b} = \frac{c}{d} \text{ Section III.} \right)$$

Example 1.—Reduce $13^\circ 12' 15''$ to the decimal of a degree, and $13^g 12' 5''$ to the decimal of a grade.

$$\begin{array}{lcl} (a) & 60 \overline{) 15} \text{ seconds of arc.} & (b) \quad 13^g 12' 5'' = 13.1205 \text{ g.} \\ & 60 \overline{) 12.25} \text{ minutes of arc.} & \\ & \underline{\underline{13.2041}} \text{ degrees of arc.} & \text{(Note.—Methods as in Section I.)} \end{array}$$

Example 2.—Convert $14^\circ 6'$ to grades.

$$\frac{14.1^\circ}{90^\circ} = \frac{x^g}{100^g}, \quad 90x = 1410.$$

$$x = 15.\dot{6}6666 \text{ grades, or } \underline{\underline{15 \text{ gr. } 66' 66\frac{2}{3}''}}.$$

Exercises 40

1. What fraction of a right angle is each of the following: 60° , 45° , 25° , 25^g , 45^g , 1° , $20'$, $36'$?
2. How many degrees are there in (a) 2 right angles, (b) $1\frac{1}{2}$ right angle, (c) $\frac{2}{3}$ right angle? Draw these angles with your protractor.

3. How many degrees are there in $\frac{1}{3}$ right angle, $\frac{1}{2}$ right angle? Express your fractions of a degree both as decimals and as minutes of arc.
4. How many minutes of arc are there in (a) 0.38° , (b) 0.25° , (c) $360''$, (d) $2^\circ 15'$, (e) 4.06° ?
5. Express 16.0875° as degrees, minutes, seconds.
6. Reduce $45^\circ 15' 4''$ to degrees and the decimal of a degree.
7. How many grades are there in 2 right angles, $1\frac{3}{4}$ right angle, 2.875 right angles?
8. Convert (a) $\frac{1}{2}$ right angle, (b) $\frac{1}{3}$ right angle, (c) $1\frac{1}{4}$ right angle into grades.
9. Express as grades, minutes, and seconds the following angles: 0.3657° , $46283'$, $.03^\circ$, 62.3807° .
10. How many grades equal 45° , 20° , 75° ?
11. How many degrees are there in 13.6° , 57.4° , 33.3° ?
12. Convert $17^\circ 14'$ to grades.
13. One angle is 40° , another 45° . Which is the larger, and by how many degrees?
14. Construct angles of 50° , 25° , 33.3° .

THE CHORD

A chord is the straight line joining any two points on the circumference of a circle. It divides a circle into two *segments*, which are only equal when the chord is the *diameter* of the circle. The chord of any circle cannot be longer than the diameter. If the extremities of a chord be joined to the centre of the circle, an angle is made as in fig. 25. Here both AB and *ab* are chords, and their difference in length is obviously due to the fact that they are chords of different circles. But *Oab* and *OAB* are similar triangles, and therefore—

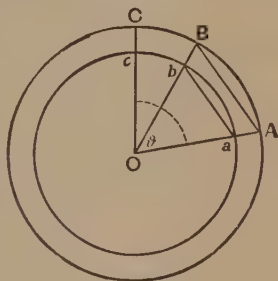


Fig. 25

$$\frac{\text{Chord } AB}{\text{Chord } ab} = \frac{\text{radius } OA}{\text{radius } oa}; \text{ ep. } \frac{a}{b} = \frac{c}{d}.$$

Again, if the angle is increased to *AOC* and the chords *CA* and *ca* drawn, the chords would still be in the same ratio as the

radii of the circles, although they would have increased in size with the angle.

We observe then—

(a) That the chord of an angle varies with the angle, i.e. it decreases or increases as the angle decreases or increases.

(b) That chords of the same angle in circles of differing magnitude bear the same ratio to one another as the radii of the circles.

Thus it would not do to say that a chord of a certain length in *any* circle would always subtend a certain angle. It is necessary to take a circle of *standard* size, and for this purpose circles of 1 unit radius are drawn, and the lengths of the chords for the various angles tabulated. The table is found at the end of this volume, and from it you read that an angle of 35° is subtended by a chord .601, and an angle 85° by a chord 1.351. The meaning of these statements is, of course, that in a circle of 1 ft. radius a chord of .601 ft. subtends an angle of 35° , or in a circle of 1 cm. radius a chord of .601 cm. subtends 35° , in a circle of 1 in. radius 85° has a chord of 1.351 in., and so on.

Chords are little used to measure angles, mainly because it is difficult to construct angles with chords, say, .601 in. or 1.351 cm. It will be obvious from fig. 25 that this difficulty can partly be overcome by taking the circle not 1 in. in radius, but, say, 5 in. Then for 35° the chord, instead of being .601 in., would be $5 \times .601$, or 3.005, and with the careful measurement of a longer line a smaller error would be made in constructing the angle. Still, even with this, the method is far from satisfactory. Again, there is no easy mode of converting a chord measure to

degrees, &c.

Example.—Construct the angle whose chord is .518.

Method.—If radius is 1 in., chord is .518 in. (just over $\frac{1}{2}$ in.). If radius is 5 in., chord is 2.590 (approx. 2.6 in.), i.e. $5 \times .518$.

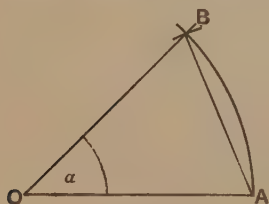


Fig. 26

Make OA (fig. 26) 5 in. From O as centre, OA as radius, strike off an arc. From A mark off AB 2.6 in. long. Join OB. AOB is approximately the angle required.

Exercises 41

1. With your protractor mark off angles of (1) 20° , (2) 60° . Make the arms of the angles 3 in. long and join their extremities. Carefully measure these lines.
2. Construct the angle whose chord is $\cdot 845$.
3. Construct an isosceles triangle whose equal sides are 4 in. and whose base is 3.1 in. What is the measure of the angle at the vertex in chords? By reference to the table read its measure in degrees.
4. Construct an equilateral triangle ABC of 3 in. side. Join A to D the middle point of BC. O is the middle point of AD. Join OB and OC and determine by measurement the magnitude of the angle BOC and its chord.
5. If the chord of an angle is quoted as $\cdot 433$, what length must the chord be in a circle of 3.5 in. radius to subtend this angle?
6. The chord of an angle is 3.38 m. when the radius of the circle is 4 m. Find the measure of the angle as stated in the tables and its equivalent in degrees.
7. From the tables I find that the chord of $54^\circ = \cdot 908$. What would be the length of the chord subtending 54° at the centre of a circle of (a) 8 in. radius, (b) 10 in. radius, (c) 7 cm. radius?
8. In a circle of 2 cm. radius the chord of an angle measures 2.828 cm. What is the angle in degrees?
9. What angles are subtended by chords measuring 1 and 2?
10. Construct an equilateral triangle of 4 in. side. Bisect each of the angles, and, using the point O of intersection of the bisectors, circumscribe a circle about the triangle. Determine the length of the radius by measurement, and the chord subtending each of the angles at O.

THE RADIAN

The radian depends on the *arc* subtending an angle at the centre of a circle, an arc equal to the radius of the circle subtending an angle of *one radian*.

Exercises 42.—Preliminary Questions

- (a). How many times can the radius of a circle be marked off *along* the circumference of a circle?
- (b). Will one radian be 60° ? Give reasons for your answer.
- (c). $\pi = 3.1416$. What number of degrees of a circle does an arc π times the radius subtend?
- (d). How many degrees is the angle subtended by an arc $\frac{\pi}{2}$ times the radius?
- (e). Given that π radians equal 180° how could you utilize the equation $\frac{a}{b} = \frac{c}{d}$ to find the number of degrees in 1 radian?

- (f). A wheel rotates once. How many degrees of arc has a point A on its circumference swept over? How many times the radius of the wheel has it gone?
- (g). A carriage wheel of 4 ft. diameter is rolled once over the ground. What number of times its radius has it gone? Give this distance in feet.

From the above it is seen that since the circumference of a circle is 2π , or 2×3.1416 its radius, the radius OA will divide

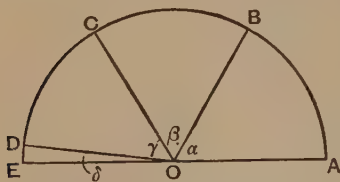


Fig. 27

into the semicircumference π times, giving the points B, C, and D in fig. 27. Since AB is an arc equal in length to OA, the radius; the angle subtended by it (i.e. angle α) is termed a radian. Thus the semicircle contains 3 whole radians and a fraction ($.1416$ radian). Thus

the semicircle contains π radians and 180° , or π radians = 180° .

The chord and the radian are similar in idea, the only difference lying in the fact that in the chord the measure of a *straight line* subtending the angle is taken, while in a radian the measure of the *arc* subtending the angle is considered. Just as in the case with chords, the greater the radius of the circle, the greater must be the arc subtending one radian.

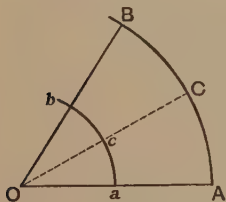


Fig. 28

In fig. 28 the arc AB equals the radius OA, and the arc ab equals the radius Oa .

The angle at O is therefore a radian, and the figures Oab and OAB are similar figures, if not similar triangles.

$$\text{Thus } \frac{\text{arc } ab}{\text{arc } AB} = \frac{\text{radius } Oa}{\text{radius } OA}; \text{ cp. } \frac{a}{b} = \frac{c}{d}$$

OC bisects the angle O. It would therefore be expected that AC was half OA and ac half Oa , for—

$$\frac{OA}{Oa} = \frac{AC}{ac}, \text{ since the figures are similar.}$$

OC therefore subtends an angle of $.5$ radian.

In the table of angles the lengths of the arcs are quoted for a radius of 1 unit. So I note that $40^\circ = .6981$ radian, or the arc subtending $.6981$ radian, or 40° , in a circle of 1 in. radius will be $.6891$ in. long, in a circle of 100 cm. radius will be 69.81 cm. long, and so on. Here again there is similarity between the radian and chord. Thus the length of the arc is always the product of the angle it subtends expressed in radians and the radius, or $A = R \times \theta$. Similarly, $\theta = \frac{A}{R}$, or the length of the arc divided by the length of the radius expresses the angle in *radians*.

An inspection of the tables of radians and chords with degrees will show one distinct superiority of the radian over the chord. You read, for example, that $45^\circ = .7854$ radian, and that $9^\circ = .1571$ radian. Now $9^\circ = \frac{1}{5}$ of 45° , and $\frac{1}{5}$ of $.7854$ radian $= .15708$, or $.1571$ to the nearest 4th figure. This cannot be done in the case of chords.

Exercises 43

- Construct an angle of 60° (1.0472 radian) at the centres of circles of radii (a) 1 cm., (b) 6 cm., (c) 5.6 in. What is the length of the arc in each case?
- How many radians are there in a semicircle? From it state in radians 90° , 45° , 60° , 30° .
- The sector of a circle (the figure bounded by two radii and the arc of a circle) of 3 in. radius has its arc 3.927 in. long. Find the angle between the two radii (a) in radians, (b) in degrees (use table).
- The arc of a sector subtends an angle of $.9599$ radian. If the radius of the arc is 3.75 in. find the length of the arc.
- The radii of two circles are 4 cm. and 7 cm., and from them sectors are taken with arcs in each case subtending an angle of 45° . Find (a) the lengths of the arcs, (b) the ratio of the arcs (use your tables).
- Given $90^\circ = 1.5708$ radian, from it determine 9° , 45° , 10° , 1° , and 4° in radians.
- Concentric circles are drawn of which sectors are drawn having a common angle of 30° . $30^\circ = .5236$ radian. What is the length of the arcs subtending this angle if the radii of the circle are 1 in., 1.5 in., 2 in., 2.5 in., and 3 in.? If the radius were r inches, and $30^\circ = \theta$ radians, what is the length of the arc? What general formula can be written for finding the arc, given the angle and radius?

Conversion of Radians to Degrees and some Practical Applications.

Example 1.—How many radians are there in 25° ?

Basis of reckoning is—

$$\pi \text{ radians} = 180^\circ$$

$$\therefore x \text{ „} = 25^\circ$$

Thus $\frac{x}{\pi} = \frac{25}{180}$; $\left(\frac{a}{b} = \frac{c}{d} \text{ Section III}\right)$.

$$x = \frac{25 \times \pi}{180} = \frac{5 \times 3.1416}{36}$$

$$= .4363 \text{ radian.}$$

$$\begin{array}{r} 36 \overline{) 15.7080} \quad (.4363) \\ \underline{144} \\ 130 \\ \underline{108} \\ 228 \\ \underline{216} \\ 120 \\ \underline{108} \\ 120 \\ \underline{120} \\ 0 \end{array}$$

Example 2.—The same line of reasoning gives the **area of the sector** of a circle as well as the angle in radians.

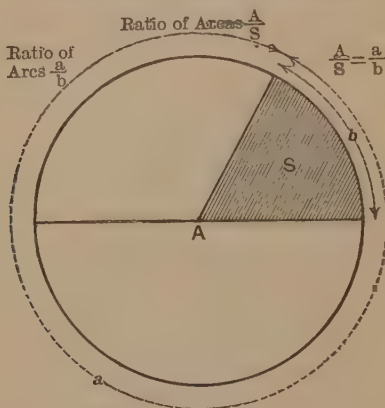


Fig. 29

Let A = area of the circle, S = area of sector, a = circumference of the circle, b = arc of the sector.

Then it follows: That the area of the circle A is just as much greater than the area of the sector S as the circumference a is greater than the arc b ; or—

$$\frac{A}{S} = \frac{a}{b} \dots\dots\dots 1.$$

But $A = \pi r^2$ (area of a circle) and $a = 2\pi r$ (circumference of circle). Substituting in 1, we get—

$$\frac{\pi r^2}{S} = \frac{2\pi r}{b}, \text{ i.e. } S = \frac{\pi r^2 \times b}{2\pi r} = \frac{r \times b}{2} \dots\dots\dots 2.$$

Thus the area S of the sector equals half the product of the radius and the arc.

Note.—This formula is akin to that of the triangle whose area equals half the product of the base and the height, the radius and arc of the sector corresponding to the perpendicular height and base of the triangle.

This formula can be further modified if the angle of the sector is given. If the number of degrees be converted to radians on the lines of Exercise 38, the arc b of the sector is the radius r multiplied by the angle of the sector in radians. Exercise 43, No. 7, gives the formula as arc $b = \theta r$ where θ = measure of angle in radians and r = radius.

Thus 2 becomes—

$$S = \frac{br}{2} = \frac{r\theta r}{2} = \frac{r^2\theta}{2} \dots\dots\dots 3.$$

I.e. the area of the sector of a circle equals half the product of the angle in radians and the square of the radius.

There is no need to remember these formulæ as they are so easily recovered from first principles. This process of reasoning is very valuable. You have grown accustomed to it in two former sections.

Example 3.—The **slant surface of a cone** is the surface of the cone other than the base. If a cut could be made along a line from the vertex to the base of a right cone, as AB in fig. 30, and the surface layer of the cone peeled off and laid flat, a *sector* of a circle would be formed. Reflect on this, and it will be soon quite clear that this must be so, since every point on the rim of

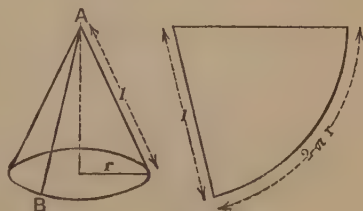


Fig. 30

quite clear that this must be so, since every point on the rim of

the base is equidistant from the vertex of a right cone. The arc of this sector will be the circumference of the base and the radius the slant height of the cone.

$$\text{Now } \frac{\text{area of sector}}{\text{area of circle}} = \frac{\text{length of arc}}{\text{circumference of circle}} \dots\dots\dots 1.$$

If l = slant height of cone, then l = radius of sector which is part of the circle of area πl^2 and of circumference $2\pi l$. Further, if r = radius of base of cone, then $2\pi r$ is the circumference of the base of the cone and length of the arc of the sector. Substituting these values in 1, we have—

$$\frac{\text{Area of sector}}{\pi l^2} = \frac{2\pi r}{2\pi l}.$$

$$\therefore \text{Area of sector} = \frac{\pi l^2 \times 2\pi r}{2\pi l} = \pi r l \dots\dots\dots 2.$$

Thus the slant surface of a cone is the product of π , the radius of the base, and the slant height of cone.

E.g., find surface area of a cone of slant height 8 in. and base diameter 4 in.

$$S = \pi r l = 3.1416 \times 2 \times 8 = 16 \times 3.1416 \dots\dots 1.$$

$$\text{Area of base} = \pi r^2 = 3.1416 \times 4 = 4 \times 3.1416 \dots\dots 2.$$

$$\begin{aligned} \text{Total surface area} &= (1) + (2) = 20 \times 3.1416 \\ &= \underline{\underline{62.832 \text{ sq. in.}}} \end{aligned}$$

Exercises 44

1. From first principles determine the degrees in 1 radian.
2. Reduce $\frac{\pi}{3}$ radians to degrees. From the result determine 10° in radians.
3. A wheel has a mark on its circumference. This mark sweeps over 16π radians. How many revolutions has the wheel made?
4. A wheel rotates (a) once, (b) twice, (c) 50 times. In each case state how many radians a point on its circumference sweeps over? If the radius of the wheel is 1 m. how far would the wheel roll in each of the three cases?
5. A wheel makes 240 revolutions per minute. (a) How many radians does any point sweep out per minute? (b) Hence state how far a point A, 1 ft. from the centre, has gone in a minute? (c) How far

has a point B 3 ft. from the centre gone in a minute? (d) What is the speed of a belt passing over this wheel if the wheel be 1 ft. 6 in. radius?

6. A wheel of 2 ft. radius makes 40 revolutions per minute. How long is any point on its surface in sweeping out 1 radian? Hence state the angular velocity of the wheel, i.e. how many radians it travels per second.
7. Find the area of the sector of a circle of 6.5 cm. radius when the arc subtends an angle of 30° .
8. The arc of a sector is 15 in. and its radius 7 in. Find its area.
9. Construct an angle of 60° and from the arms cut off portions 2.5 in. long. What is the area of the sector so formed? Draw the chord of the arc and by measurement determine the area of the triangle made up of the chord and two radii. What is the area of the *segment* of the circle cut from the sector?
10. In a circle of 3.25 in. radius construct a regular hexagon. Join the centre O with the angular points. What is the area of each segment and sector so formed?
11. The extremities of an arc of 16 in. are joined to the centre of the circle whose diameter is 18 in. What is the area of the sector enclosed?
12. Find the area of the slant surface of a cone whose slant height measures 7.5 in., and the diameter of whose base is 6.4 in.
13. If the total surface area of a cone is 60.4758 sq. in., and the radius of its circular base is 3.5, find its slant height.
14. The radius of the sector of a circle is 6.75 in. and its area is 101.25 sq. in. Find the length of the arc and the measure of the angle it subtends in radians.
15. The radius of the base of a cone is 8.8 cm. Its slant height is 176 cm. Determine (a) area of the base, (b) area of the slant surface, (c) total surface area.

PERPENDICULARS AND BASES OF RIGHT- ANGLED TRIANGLES

Consider carefully fig. 31, and satisfy yourself as to the facts elucidated by the questions put concerning it.

OAB is a right-angled triangle, right-angled at B, and having at O the angle θ which is to be considered. The side opposite or subtending this angle is AB, and is called for the sake of convenience the "Perpendicular" to the angle, while the side OB adjacent to the angle is called its "Base". The side OA opposite the right angle is generally termed the hypotenuse.

in the triangle AOB as the angle θ is increased from 0° to 90° ; and (2) that the ratio of the perpendicular to the hypotenuse gradually increases while the ratio of the base to the hypotenuse gradually decreases as the angle increases from 0° to 90° .

The sine of an angle is the ratio of its perpendicular to the hypotenuse, that is $\frac{p}{h}$ (fig. 31). It may also be considered to be the length of the perpendicular when the hypotenuse is one unit long, for $\frac{p}{h}$ when $h = 1$ equals p . The *sine* of 30° is written shortly " $\sin 30^\circ$ ", and is found from the table at the end of the book to be $.5$. It is written thus: $\sin 30^\circ = .5$. An angle of 30° can be constructed from its *sine*. Since the *sine* is the ratio $\frac{p}{h}$ (where p = perpendicular and h = hypotenuse), it follows that if—

$$\sin 30^\circ = .5$$

$$\therefore \frac{p}{h} = .5 \text{ or } \frac{.5}{1},$$

that is the perp. (p) = $.5$ when the hypot. (h) = 1 .

If then a triangle OAB be constructed, right-angled at A, such that the perpendicular AB is $.5$ in. and the hypotenuse OB is 1 in. (i.e. the perpendicular made *half* the hypotenuse) the angle at O is 30° .

The cosine of an angle is the ratio of its base to the hypotenuse, that is $\frac{b}{h}$ (fig. 31). It may also be considered to be the length of the base if the hypotenuse is made of unit length. The cosine of 45° is written briefly " $\cos 45^\circ$ ", and its value can be found in the tables, $.7071$. It is written thus $\cos 45^\circ = .7071$. From this an angle of 45° can be constructed. Make OAB a right-angled triangle with its hypotenuse, say, 5 in. long, and its base OA $5 \times .7071$ in. long. Then the angle at O is 45° , for the ratio of its base to its hypotenuse is $\frac{5 \times .7071}{5} = \frac{.7071}{1} = .7071$, which is " $\cos 45^\circ$ ".

The tables show that while the sines of angles gradually increase from 0 to 1 as the angle increases from 0° to 90° , the

cosines gradually decrease from 1 to 0 as the angle so increases
e.g.—

Degrees.	Sines.	Cosines.
0°	$\sin 0^\circ = 0$	$\cos 0^\circ = 1.0$
30°	$\sin 30^\circ = .5$	$\cos 30^\circ = .866$
45°	$\sin 45^\circ = .7071$	$\cos 45^\circ = .7071$
60°	$\sin 60^\circ = .866$	$\cos 60^\circ = .5$
90°	$\sin 90^\circ = 1.0$	$\cos 90^\circ = 0$

Fig. 31 will bear this out if the angle θ be made successively 0°, 30°, 45°, 60°, and 90°.

Exercises 45.—Geometrical

- Construct an angle of 30°. Call it BAC. On AC mark off distances of 1 in., 1.5 in., 2 in., 2.5 in., and 3 in., and let fall perpendiculars from these points upon AB. Carefully measure the perpendiculars so obtained. What is the ratio of each perpendicular to its hypotenuse? What is the value of $\sin 30^\circ$? What is the length of the perpendicular when the hypotenuse is 1 in., 1 cm., 1 m.?
- Construct an angle of 60°. Call it BAC and proceed as in Question 1. From your working determine $\sin 60^\circ$ as accurately as you can.
- Construct an angle of 60° and call it AOC. On OA as the base line mark off distances of 1 in., 2 in., and 2.5 in., and from them raise perpendiculars to meet OC or OC produced. Carefully measure the hypotenuses of the right-angled triangles so formed, and state the ratio of the base to the hypotenuse in each case. What is $\cos 60^\circ$?
- A kite string makes 30° with the ground and is 500 ft. long. Assuming that the string is taut, draw a figure to represent the position of the kite, and determine how high the kite is above the ground.
- Construct a 45° right-angled triangle and make it as large as possible. Carefully measure with your ruler the perpendicular, base, and hypotenuse, and determine therefrom $\sin 45^\circ$ (i.e. $\frac{\text{perp.}}{\text{hypot.}}$) and $\cos 45^\circ$ (i.e. $\frac{\text{base}}{\text{hypot.}}$). Compare your answers with one another and with the values given in your tables.
- What is the sine of an angle? Given that the sine of an angle is .8 construct the angle.
- What is the cosine of an angle? By construction determine $\cos 15^\circ$.
- A ladder 50 ft. long makes 75° with the ground and just reaches the top of a wall. How high is the wall and how far is the foot of the ladder from the foot of the wall? (Draw a figure and use your tables.)

9. Every angle of an equilateral triangle is equal to 60° , and $\sin 60^\circ = .866$. What is the altitude of an equilateral triangle whose side measures 5 in.? Find your answer from the value of $\sin 60^\circ$ and check your result by measurement.

Application of Sine and Cosine to Practical Work.

Example.—Two sides of a triangle ABC are 8 in. and 9.3 in. long, and the angle between them is 40° . Find its area.

Method—

$$A = \frac{1}{2}b \times h, \text{ where } b = \text{base and } h = \text{height.}$$

$$A = \frac{1}{2} \times 9.3 \times h. \quad (\text{Cf. Section II.})$$

This h (height or altitude) has first to be found before the sum can be completed.

In fig. 32 h is the perpendicular of a right-angled triangle ADB whose hypotenuse is 8 in., and whose angle of reference is 40° .

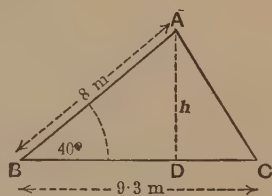


Fig. 32

$$\text{From fig. 32, } \sin 40^\circ = \frac{h}{8},$$

form the table of angles

$$\sin 40^\circ = .6428 \text{ or } \frac{.6428}{1},$$

$$\text{i.e. } \frac{h}{8} = \frac{.6428}{1} \quad \left(\frac{a}{b} = \frac{c}{d} \quad \text{Section III.} \right).$$

$$h = 8 \times .6428 = 5.1424.$$

$$\begin{array}{r} 9.3 \\ 2.5712 \\ \hline 18.6 \\ 4 \ 65 \\ 65 \\ 1 \\ \hline 13.91 \end{array}$$

This gives h .

$$\therefore A = \frac{1}{2} \times 9.3 \times 5.1424 = \underline{\underline{13.91 \text{ sq. in.}}}$$

Exercises 46

(Use tables of sines and cosines)

1. A triangular plate has one of its angles 45° and the shorter side adjacent to it 1 ft. long. Determine the altitude of the triangle.
2. An equilateral triangle (each angle is 60°) has each of its sides 3 in. long. Find the altitude of the triangle.
3. A parallelogram has its adjacent sides 3.65 and 8.7 cm. long, and the angle included between 50° . Find the width of the parallelogram and its area.

4. ABCD is a parallelogram. AB is 6.5 cm. BC is 5.6 cm., and the angle at B is 30° . From C drop a perpendicular to AB. Find its length and so determine the area of the parallelogram.
5. ABC is a right-angled triangle. A is the right angle, and C may be any angle constructed carefully with the protractor. By carefully measuring the hypotenuse, perpendicular, and base, show that—
 - (a) $\sin C \times \text{length of BC} = \text{length of the perpendicular AB}$.
 - (b) $\cos C \times \text{length of BC} = \text{length of the base AC}$.
6. In a right-angled triangle ABC, right-angled at A, the hypotenuse BC is 10 cm. long. Find the length of the base AC when the angle C is 20° , 40° , 50° , 80° .
7. ABC is a right-angled triangle. Angle A is the right angle and angle B is 30° . If BC measures 10 in., 10 cm., or 10 hf.-in., show that—
 - (a) $10 \sin B = 5 \text{ in.}, 5 \text{ cm.}, \text{ or } 5 \text{ hf.-in.}$ (i.e. length of AC).
 - (b) $10 \cos B = 8.6 \text{ in.}, 8.6 \text{ cm.}, \text{ or } 4.3 \text{ in.}$ (i.e. length of AB).
 If BC measures 8 in., 4 in., 3 cm., determine the lengths of AC and AB in each case.
8. The altitude of a triangle subtending an angle of 65° is 7.5 in. Determine the length of one side of the triangle.
9. Find the area of a triangle whose adjacent sides measure 15.68 cm. and 12.35 cm. and include an angle of 50° .
10. The altitude of a triangle is 12 cm., and it subtends two angles of the triangle one 65° and the other 28° . Determine the lengths of the three sides by using your tables of sines and cosines. (Note the segments of the base can be determined by means of cosines, after the hypotenuses of the two right-angled triangles formed by the perpendicular from the vertex have been found from the sines.)
11. Given that $\cos \theta = .6$ construct the angle and find by measurement $\sin \theta$.
12. $\sin \theta = \frac{D + d}{2C}$. What is the angle θ if $D = 16$ $d = 4$ and $C = 20$?
13. The formula for the diagonal D of a parallelogram whose adjacent sides are a and b and where a and b include any angle, say θ , is given thus: $D^2 = a^2 + b^2 + 2ab \cos \theta$. Find D when the sides of the parallelogram measure 3 in. and 4 in. and include an angle of 60° .
14. Construct a triangle of sides 5.8 in., 3.6 in., and 4.5 in. By drawing perpendiculars from the angular points in turn to the opposite sides, and carefully measuring them, determine by sines and cosines the measures of the three angles, each to the nearest degree.

SOME IMPORTANT TRUTHS ABOUT TRIANGLES

Perform carefully the following experiments:—

Exercises 47**A**

1. Construct any triangle ABC. With your protractor carefully measure each of the three angles. What is the sum of these angles in (a) degrees, (b) right angles, (c) radians?
2. The triangle ABC has the angle A 30° and the angle B 70° . What is the size of the angle C?
3. The side AB of any triangle ABC is produced to D, and from B a line BE parallel to AC is drawn. There are three angles at B. What do they equal when added together? Express them in terms of (a) right angles, (b) the three angles of the triangle ABC. What angles of the triangle are together equal to the exterior angle CBD?
4. On AB 2.5 in. long construct a semicircle. Take any point C on the arc, and, joining CA and CB, make a triangle ABC. What is the size of angle C in degrees? Take any other point and repeat the work. Does the angle at C vary in size?
5. ABC is any triangle, right-angled at A. Carefully measure the angles B and C and state what number of degrees is obtained when they are added together. Perform experiments on three different triangles. Is there any difference in your results? If angle B were 30° what is angle C, and what is B if C is 70° ? B and C are called "complementary angles". Why?

B

1. Construct a triangle having its sides equal respectively to 5 cm., 4 cm., and 4.5 cm. Which side of the triangle subtends (a) the largest angle, (b) the smallest angle?
2. On AB, 3 in. long, construct a triangle ABC having the angle A 60° and the angle B 45° . What is the value of angle C? After carefully measuring the sides state your observations of the lengths of these sides in connection with the sizes of the three angles of the triangle.
3. BAC is a right angle, on BA mark off distances AD 1 in. and AB 2 in. Make AC 2 in. long. Join CD and CB. In the triangle BCD why is BC longer than BD? How do you know by inspection that the angle opposite the side BD is the smallest one in the triangle? The angle at D is obtuse, which angles of the triangle ABD must be added together to make D?

The foregoing exercises lead to a better understanding of the following statements:—

(a) The angles of any triangle are together equal to two right angles.

(b) The exterior angle of a triangle is equal to the sum of the two interior opposite angles.

(c) In a right-angled triangle the two acute angles (together) equal one right angle.

(d) The angle to be added to any acute angle to make a right angle is called its "complement". Similarly, "supplementary angles" are such that equal two right angles when added together.

(e) The longest side of a triangle has the largest angle opposite it, and vice versa. Similarly, the shortest side is always opposite the smallest angle.

(f) The angle in a semicircle is a right angle.

TANGENTS

A tangent is a straight line touching the circumference of a circle without cutting it. A tangent is always at right angles to the radius of the circle.

The geometrical construction of tangents depends upon two facts, viz.:—

1. A tangent and a radius are perpendicular to one another, and
2. The angle in a semicircle is a right angle.

By joining the centre of the circle with the point from which the tangent originates, and on it making a semicircle, a radius and tangent can be drawn making an angle in the semicircle.

A. To draw a tangent to a circle from a given point outside.

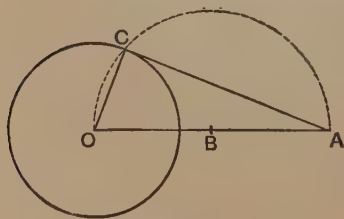


Fig. 33

Construction.—Join O , the centre of the circle, with the point A .

On OA construct the semicircle OCA .

From C , where the circumference of the circle cuts the semicircular arc, draw OC and CA .

Then CA is the tangent.

Reason for construction.—The lines joining O and A to any point on the arc of the semicircle make a right angle. Thus OC (the radius) and CA are at right angles to one another, and CA is the tangent required.

B. To draw a tangent touching two unequal circles (externally).

Construction.—Join O and A, the centres of the circles.

Mark off DC equal to the radius of the smaller circle, and describe the dotted circle.

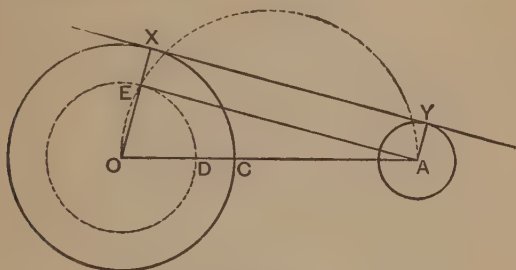


Fig. 34

Draw a tangent from the point A to the dotted circle, just as in the above case (A).

The tangent XY is parallel to the constructed tangent EA, and can be drawn by producing the radius OE to X.

Reasons for construction.—1. OE and EA are at right angles to one another, because E is a point on a semicircumference.

2. The dotted circle is just as far from the circumference of the larger circle as A is from the circumference of the smaller circle.

C. To draw a tangent touching two unequal circles (internally).

Construction.—Join O and A, the centres of the circles.

Mark off DE (outside the larger circle) equal to the radius of the smaller circle.

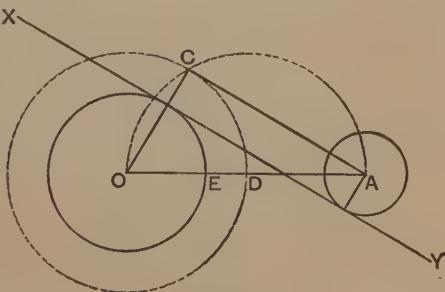


Fig. 35

Draw a tangent AC to the dotted circle through O (see A, above).

The tangent XY is parallel to CA, and is drawn through the point where OC intersects the circumference of the larger circle.

Reasons for the construction.—1. OC and CA are perpendicular to one another, since C is a point on the semicircumference OCA.

2. The dotted circumference is just as far outside the larger circle as the point A is within the smaller circle.

Exercises 48

1. From a point 1.5 from the circumference of a circle of 2 in. radius, draw a tangent to the circle. How many tangents to this circle can be drawn from this point?
2. A point A is 4 cm. from the centre of a circle of 2.5 cm. radius. Draw tangents from point A to the circle.
3. Two circles of 3 cm. and 2 cm. radius have their centres 7 cm. apart. Draw an external tangent touching these two circles.
4. Two circles of 3 in. and 2 in. diameter have their centres 3 in. apart. Draw an internal tangent touching these two circles.
5. BAC is an angle of 30° . Bisect it by the line AD. From any point X on AD perpendiculars are drawn to AB and AC. These perpendiculars are equal, and with X as centre and one perpendicular as radius a circle may be drawn touching the sides AB and AC. Why are the perpendiculars equal? Why is each arm of the angle a tangent to the circle?
6. ABC is any triangle, and two of the angles A and B are bisected. Let the bisectors meet at O. How far is O from the three sides of the triangle? A perpendicular is drawn from O to AB, and with O as centre and AB as radius a circle is drawn. Which sides of the triangle are tangents to the circle?

The Similar Triangles in a Semicircle.

Exercises 49

1. ABC is a semicircle, and from any point C on the semicircumference CA, CB are drawn. From C a perpendicular CD is drawn to AB. Construct such a figure, making AB 6 in. long. Carefully cut out the triangles ADC and CDB and compare their angles. What observations can you make?
2. In fig. 36 there are three similar triangles; name them.
3. What angle in the triangle CDB is equal to the angle A (marked θ)?

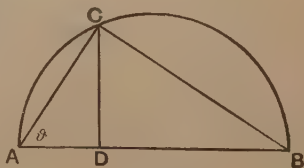


Fig. 36

4. It is said that AD bears the same relationship to DC that DC does to DB. How can you justify this statement?
5. If $\frac{AD}{CD} = \frac{CD}{DB}$ what special name can be given to CD?
6. On a line AB, 8 cm. long, construct a semicircle. At A make an angle of 60° , and where the arm AC cuts the semicircumference, join C to B. ACB is a right-angled triangle. From C drop CD perpendicular to AB. (a) Name the similar triangles in the figure. (b) Carefully measure the lengths of AD, CD, and BD with a ruler, and express the ratio $\frac{AD}{CD}$ and $\frac{CD}{BD}$. What do you observe about these ratios? (c) How can you use this figure to graphically determine the mean proportional between 2 cm. and 6 cm.? (d) If you had a rectangle whose sides measured 2 cm. and 6 cm., what is the length of the side of a square equal in area to the rectangle?
7. A line AB 10 cm. long is divided at C so that AC is 2 cm. long. On AB a semicircle is constructed, and from C a line CD is drawn perpendicular to AB to cut the arc in D. By joining DA and DB fig. 36 is produced. What is the length of CD, and what is the proportional between 2 and 8?
8. A rectangle is constructed with its sides 2 and 8 cm. respectively, and a square equal in area to the rectangle. What is the length of the side of the square, and the relationship existing between the side of the square and the sides of the rectangle?
9. Find graphically (i.e. by fig. 36) the mean proportional between 4 and 9.

Fig. 36, with the exercise on it, should be considered with a revision of the last exercise in Section III. The perpendicular CD always divides the right-angled triangle into two triangles *similar* to the original triangle, and if the similarly placed sides are considered it will be found that CD is the mean proportional of the segments of AB, for $\frac{AD}{CD} = \frac{CD}{DB}$. The figure also furnishes a means of making a square equal to a given rectangle; for if AD and DB are the sides of the rectangle CD is the side of the required square. Since

$$\frac{AD}{DC} = \frac{DC}{DB}, \text{ then } DC \times DC = AD \times DB$$

(by cross multiplication).

If, then, AD and DB are the two sides of a rectangle, its area is $AD \times DB$, and the square on DC is equal to it.

The Tangent of an Angle.—Refer to fig. 24 for the following:—

PR is the tangent to the circle which was then considered, and OP is the radius. The triangles OPK and OQP are similar (compare fig. 36).

Therefore $\frac{PR}{OP} = \frac{PQ}{OQ}$. Or, in other words, the relationship between the tangent and the radius is equal to the relationship between the perpendicular and the base of the triangle OPQ.

In fig. 31 the ratio $\frac{p}{h}$ was termed $\sin \theta$ (the sine of the angle θ), and the ratio $\frac{b}{h}$ was termed $\cos \theta$ (the cosine of the angle θ).

Now the ratio $\frac{p}{b}$ is to be considered. It is termed the *tangent* of the angle θ (written $\tan \theta$). The ratio of the perpendicular to the base is so named because this ratio is equal to the ratio of a tangent from a point to the radius from its point of contact, as we have just discovered.

When discussing fig. 24 we found that as the angle θ increased, the tangent grew longer, while if it decreased, the tangent grew smaller. Thus the ratio $\frac{p}{b}$ increases and decreases with the angle θ , and this is tabulated for the angles from 0° to 90° in the table of angles at the end of the book. Thus $\tan 60^\circ = 1.732$, which means that in a right-angled triangle of perpendicular 1.732 in. and base 1 in. the angle of reference is 60° .

Example.—Construct an angle whose tangent is one ($\tan \theta = 1$). Here the ratio of perpendicular to base is $\frac{1}{1}$, i.e. the base equals the perpendicular. Thus if a triangle OAB, right-angled at O has OA equal to OB, the angle A (or B) is 45° .

Exercises 50

1. AB is a line 2.5 in. long. On it is constructed a triangle, right-angled at C (cp. fig. 36) with CA 1.5 in. long. What is the length of BC? Determine this by drawing the figure. Carefully state the sine, cosine, and tangent of the angles A and B.
2. The shadow thrown of a vertical rod 1 m. long is 1.7 m. Find the altitude of the sun by drawing a figure to scale and using your

protractor. What is the tangent of this angle? *Note.*—The altitude of the sun is the angle its rays make with the horizontal.

3. $\tan 60^\circ = 1.732$. What does this mean? Find the base of a 60° right-angled triangle whose altitude is 2.598 in.
4. The length of the shadow of a 1.5-ft. rod at midday is .75 ft., and at the same time the length of the shadow of a tree is found to be 35 ft. Find the altitude of the sun and the height of the tree.
5. A man standing 40 ft. from the base of a tree finds that the tree subtends an angle of 50° at his eye. Draw a 50° right-angled triangle whose perpendicular shall represent the tree, and whose base shall represent the distance of the man from its foot, and by measurement determine the height of the tree. Work also by use of tables.
6. A tower subtends an angle of 68° at the eye of an observer 60 ft. from its base. Find its height. Proceed thus: Draw a 68° right-angled triangle to represent the facts. Calling x the height of the tower, $\tan 68^\circ = \frac{x}{60}$ (from the figure) and $\tan 68^\circ =$ a certain value given in the tables. $\therefore \frac{x}{60} =$ the value of the tan in the tables. This is like $\frac{a}{b} = \frac{c}{d}$.
7. A man plants a vertical rod opposite a tree on the other bank of a river. He then walks 100 yd. along the bank, and, looking back, finds the line joining the rod and the tree subtends an angle of 32° . Find the width of the river. (Proceed as in 6.)
8. The two sides of a right-angled triangle are 5.1 and 6.8 cm. Find to the nearest degree, and by use of your tables, the angle subtending each of these sides, and the area of the triangle. *Note.*—The hypotenuse is not given.
9. Two vessels leave harbour at the same time. One travels due west at 16 m. per hour; the other travels south-west. At the end of $1\frac{1}{2}$ hour they are due north and south of one another. How far are they apart?
10. V , the vertical component of the earth's magnetic force, divided by H , the horizontal component, is the tangent of the angle of dip. This is given by the formula $\frac{V}{H} = \tan \theta$ or $V = H \tan \theta$. Find V when $H = 0.18$ and $\theta = 67^\circ$.

Exercises 51.—General Exercise on Use of Tables

1. From your tables state—

(1) In degrees .4189 radian, .7854 radian, 1.1345 radian.

(2) θ in degrees when $\tan \theta = 1.4281$, $\sin \theta = .3420$, cosine $\theta = .5878$.

2. *Note.*— $\sin 45^\circ = .7071$; this is often expressed thus: $45^\circ = \sin^{-1}.7071$.

Thus $\sin^{-1}.7071$ means "The angle whose sine is .7071". Deter-

mine the angles represented by $\sin^{-1} \cdot 6428$, $\cos^{-1} \cdot 9063$, $\tan^{-1} 1 \cdot 0355$, $\cos^{-1} \cdot 6428$.

3. Write out the sine, cosine, and tangent of 35° . What length will the perpendicular and base of a 35° right-angled triangle be if the hypotenuse is (a) 10 cm., (b) 7 in.?
4. By substituting the values of the angles given, prove that—
 $\cos(40^\circ + 30^\circ) = \cos 40^\circ \cos 30^\circ - \sin 40^\circ \sin 30^\circ$.
 (Note.— $\cos 40^\circ \cos 30^\circ$ means $\cos 40^\circ \times \cos 30^\circ$ just as ab means $a \times b$.)
5. Given that $\cos^2 \theta$ means $\cos \theta \times \cos \theta$, find by substitution the value to the nearest unit of $\cos^2 65^\circ + \sin^2 65^\circ$.
6. Divide $\sin 60^\circ$ by $\cos 60^\circ$. What does your answer equal? What other ratio have you found?
7. Two forces of 7 and 12 lb. act at an angle of 40° to one another. The single force R , which can be substituted for these two forces (F_1 and F_2) is given by the formula.

$$R^2 = F_1^2 + F_2^2 + 2FF_2 \cos \theta.$$

Find R .

8. ABC is a triangle. AB is 6.5 cm., BC is 5.4 cm., and the angle ABC is 50° . Determine its area.
9. The altitude of a triangle is given by the statement $l \sin \theta$ where l is the length of a side and θ the angle subtended by the altitude. Where l is 27.5 and θ is 45° find the altitude.
10. ABC is a triangle. A is 90° , B 37° , and AB is 5.32 cm. Find the other sides by using your tables of angles, and determine the area of the triangle.
11. The top of a cliff known to be 800 ft. is found to subtend an angle of 20° to an observer at sea. How far is the vessel from shore?
12. By substituting values for the following, prove—
 (a) $\sin^2 45^\circ + \cos^2 45^\circ = 1$.
 (b) $\sin(15^\circ + 18^\circ) = \cos 15^\circ \sin 18^\circ + \cos 18^\circ \sin 15^\circ$.

SUMMARY

An **angle**, the inclination of two lines, may be measured in several ways. These ways have not been unified, chiefly because each method is valuable for certain practical purposes. The ordinary method of measurement is (in Great Britain) in **degrees** and (in France) in **grades**. The degree is subdivided into 60 *min.* and each minute into 60 *sec.*; 90 deg. make a right angle. One hundred *grades* make a right angle, and each is subdivided into 100 *min.* of arc, which are each further subdivided into 100 *sec.* The conversion of grades to degrees is based on the principles of Section III.

The **chord** of a circle is also the measure of the angle it subtends at the centre of the circle, and conversion tables have been drawn up in which the length of the chord is given when the radius of the circle is unity.

A **radian** is an angle at the centre of a circle subtended by an arc equal in length to the radius. In a semicircle there are π radians. Thus π radians = 2 right angles = 180° . The relationship existing between the radius of a circle, the arc of a circle, and the angle in radians subtended by the arc is expressed by the formula $\theta = \frac{a}{r}$, where θ = measure of angle in radians, r = radius, and a = arc of circle. Conversion tables of degrees and radians have been compiled, the length of the arc of a circle of unit radius subtending the angle stated in degrees being tabulated.

Sines, cosines, and tangents are the ratios of the sides of a right-angled triangle. The ratio of the perpendicular to the hypotenuse, or $\frac{p}{h}$, is termed the sine of the angle ($\sin \theta$), the ratio of base to hypotenuse, or $\frac{b}{h}$, is termed the cosine ($\cos \theta$), and the ratio of the perpendicular to the base, or $\frac{p}{b}$, is termed the tangent ($\tan \theta$). These three ratios form the basis of trigonometry. They vary only with the size of the angle of reference.

Formulæ obtained in Section IV are:—

Area of the Sector of a Circle: $A = \frac{1}{2}rb$, where r = radius and b = length of arc.

Area of Parallelogram: $A = ab \sin \theta$, where a and b are adjacent sides and θ the angle included by them.

Area of Triangle: $A = \frac{1}{2}ab \sin \theta$, where a , b , and θ are two sides and the included angle.

The **angle in a semicircle** is a right angle. By taking any point upon the semicircumference, and from it dropping a perpendicular upon the diameter, a line is obtained whose length is the mean proportional of the lengths of the segments into which it divides the diameter. It follows, then, that the square constructed on the perpendicular is equal in area to the rectangle whose sides are equal to the segments of the diameter.

SECTION V.—PLUS AND MINUS

Signs Plus and Minus.—The signs $+$ (plus) and $-$ (minus) are placed before **quantities** to indicate their **quality**. A sign should be placed before every *term*, but so far only plus or positive quantities have been considered, and, with this understood, attention has been directed only to the *size* or *magnitude* of the quantity. Now that negative or minus quantities have to be considered, it is necessary to distinguish them in some way from the plus quantities. A positive quantity may be written with or without the plus sign, e.g. 3 m. or $+ 3$ m., but a negative quantity always has the minus sign, e.g. $- 6$ cm.

WHAT THE SIGNS $+$ AND $-$ STAND FOR

These signs have in arithmetic been loosely used to imply addition and subtraction, with success as far as that branch of mathematics is concerned, since the quantities used in arithmetic are mostly positive or plus quantities. But it is well for our purpose to entirely put aside this idea. The operations of addition and subtraction may result from the use of plus or minus signs respectively; the signs, however, can hardly be made to stand for these operations, since they belong to the terms themselves.

Thus $6.8 - 3.7$ is often considered to mean “subtract 3.7 from 6.8”. The expression really means that there are two terms (*a*) 6.8 which is positive or plus, and (*b*) 3.7 which is minus or negative. Of course the result of *combining* these two terms is 3.1, a plus term and minus term being combined by the operation of subtraction. You will say that this comes to the same thing. True, but in certain work in mathematics difficulty is found unless the real meanings of plus and minus are considered. Therefore consider these signs as attached to terms to show their quality, i.e. whether they are positive or negative. The following cases will make clearer the meanings of these signs.

Case 1. Positive and Negative Direction.—Take a point O

as the starting-point of reckoning, the origin or initial position in considering lengths.

Through O a straight line is drawn indefinite in length to left and right.

We will agree to mark off *positive directions* to the right and *negative directions* to the left.

Now a man in a boat can pull, say, 5 m. an hour on a stream which travels at the rate of 3 m. per hour. It is possible by the straight line through O to show where the boat is at the end of an hour when the man pulls (a) with the stream and (b) against it.

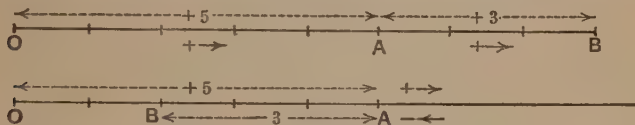


Fig. 37

When the man pulls with the stream he pulls $+5$ m. (OA, fig. 37), and the stream takes him $+3$ m. besides (AB, fig. 37). He is therefore at B at the end of an hour or $+8$ m. from O, his starting-point.

But when he pulls against the stream he pulls $+5$ m., but all the while the stream is carrying him "back 3 m." or -3 m. Thus he is at B at the end of the hour, i.e. $+2$ m. from O. The stream obviously has carried him in a direction *negative* to his own direction of progress.

This shows that the result of combining $+5 + 3 = +8$ (addition), while the result of combining $+5 - 3 = +2$ (subtraction).

It is also easy to see that if the stream's rate was 5 m. instead of 3 m. per hour against him, or -5 m. per hour, his position

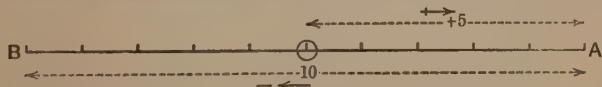


Fig. 38

would not have changed, for it would have taken him all his energy to keep the boat at O. If, on the other hand, the stream flowed at the rate of 10 m. per hour against him, instead of going

forward the stream would carry him back. This is shown in fig. 38.

The man pulls + 5 m. (OA) in the hour, but the stream has taken him - 10 m. (AB). He is therefore - 5 m. from O, i.e. 5 m. *down* the stream, 5 m. in a minus or negative direction.

This shows that the result of combining $+5 - 10 = -5$ (subtraction, but the result is minus).

Example A.— $2.5 + 4.2 - 6.2 + 1.6 - 3.1$.

Here there are 3 plus quantities and 2 negative quantities, and fig. 39 shows the result of combining these graphically.



Fig. 39

OE is the result and equals -1, and this result is obtained whichever way the terms are combined. Fig. 39 is only one way of combining them; you may try others, as, for example, marking off the negative quantities first and the positive quantities after. This fig. shows the best method of working such a sum, viz.:—

Collect the positive quantities, e.g.—

$$+ 2.5 + 4.2 + 1.6 = + 8.3 \text{ (fig. 39, OC).}$$

Collect the negative quantities, e.g.—

$$- 6.2 - 3.1 = - 9.3 \text{ (fig. 39, CE).}$$

Subtract one from other. The result has the sign of the greater $= 8.3 - 9.3 = -1$ (OE, fig. 39).

Case 2. Positive and Negative Angles.—In considering angles, (*vide*, fig. 40), the line OB is considered fixed, and the line OC supposed to be hinged at O and free to move into the positions OD, OG, OK by travelling anti-clockwise (i.e. in a direction opposite to that of the hands of a clock); or into the position OE or OH by passing over OB (i.e. clockwise). When OC travels *anti-clockwise* it is said to be travelling in a *positive* direction, and the angle it makes with OB is said to be *positive*, but when it travels *clockwise* it is moving *negatively*, and sweeps over a *negative* angle. If in fig. 40, BOC is 45° and COD is 30° , then

BOD or $+75^\circ$ is the result of combining $+45^\circ$ with $+30^\circ$. The result of combining $+75^\circ - 30^\circ$ is $+45^\circ$, for the radius has first moved in a positive direction over $+75^\circ$ to OD, and then returns in a negative direction over -30° to OC. It will be seen that the same idea is given to the angles as was given to the directions from O in Case 1, and that plus and minus signs are applied to angles as well as to lines.

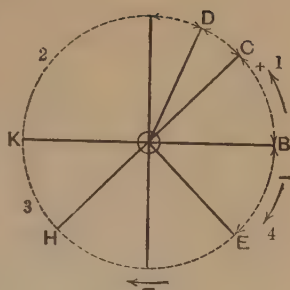


Fig. 40

Case 3. Positive and Negative Wealth.—A man's possessions in cash value must be combined with his debts if the man's wealth is to be found. The possessions are positive quantities, his debts are against him and represent negative quantities. If his assets are greater than his debts or liabilities his wealth is positive, and he is solvent; if less, he is insolvent, and he possesses as it were *negative wealth*. Thus a positive quantity in the light of bookkeeping is a *balance on the right side*, and a negative quantity a *balance on the wrong side* of the financial statement.

Case 4. Graphs show Positive and Negative Quantities.

Example.—Plot x and y for the following values, and join the points so obtained.

x	1	3	-1	-2	-4	-6
y	7	11	3	1	-3	-7

A is (1, 7), i.e. $+1$ in direction OX and $+7$ in direction OY.

C is (-1, 3) i.e. -1 " " $+3$ " "

F is (-6, -7) i.e. -6 " " -7 " "

Thus measuring to the right along OX from O is measuring in a $+$ direction, and from O to the left is measuring in a $-$ direction.

Similarly, measuring from O to Y is measuring in a positive direction, but from O to $-Y$ is measuring in a negative direction.

Since A is $(+1, +7)$ —both plus—it falls in the first angle (1 in fig. 41). Since C is $(-1, +3)$ — x minus and y plus—it

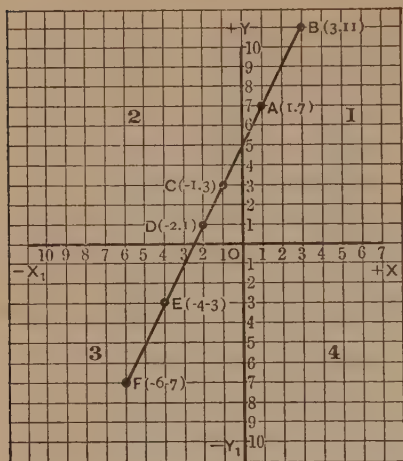


Fig. 41

falls in the second angle (2 in fig. 41). A point in the third angle (3 in fig. 41) must have both quantities negative as F $(-6, -7)$, while to get a point in the fourth angle x is positive and y negative. Thus $(6, -7)$, $(3, -6)$, $(4, -8)$ are *ordinates* of points in the fourth angle.

Case 5.—The terms “plus”, “minus”, “positive”, and “negative” are used to distinguish quantities and kinds of electricity and magnetism which are opposite

in character and behaviour. Thus we speak of “4 positive units of electricity” and “a charge of negative electricity equal to 2”, i.e. -2 units of electricity, &c. In magnetism these opposite qualities or kinds of magnetism are spoken of as “north pole” and “south pole” magnetism.

Exercises 52

1. Represent graphically and give the results of the following:—

- $+3.5$ in. $+1.6$ in.
- $+7.3$ in. -4.2 in.
- $+1.5$ in. $+3.6$ in. -5.2 in.
- $+5$ cm. $+3.7$ cm. -2.5 cm. -4.7 cm.
- -8 cm. $+5$ cm. $+6.3$ cm. -2.1 cm.
- -4.3 cm. $+8.5$ cm. -5.6 cm. $+2.3$ cm.

2. Draw angles representing:—

- $+60^\circ + 45^\circ$; (b) $+45^\circ - 30^\circ$; (c) $180^\circ + 60^\circ$ and $180^\circ - 60^\circ$.

3. A boat is pulled on a river at the rate of 12 m. per hour, and the current of the river flows at the rate of 5 m. per hour. How

- far will the boat travel up stream in 3 h.? How long will it take to return to its starting-point?
4. Two trains, A and B, are moving (a) in opposite directions, and (b) in the same direction at the rates of 30 m. and 40 m. per hour respectively. Show by diagram how far one train moves in respect of the other in one hour.
 5. Two trains, A and B, are travelling in opposite directions at 30 and 25 m. per hour respectively. At what rate does one train pass the other? If one train becomes stationary, what must be the speed of the other to pass the first train at the former rate?
 6. A ship going at a speed of 20 knots per hour is (a) aided (b) hindered by a current travelling in the same course as the ship at 5 knots per hour. How far will the ship have steamed in each case during a day of 24 h.?
 7. Represent on squared paper the following points: (3, 6), (-8, 4), (-6, -7), (5, -5). Join them by 4 straight lines; what figure is formed, and what is its area?
 8. Join the following points, A, B, and C, and find the area so formed, on the assumption that each square represents 1 sq. in. A is the point (4, 8), B (-6, -6), C (+9, -6). What is the length of the base BC?
 9. Join two points (8, 4) and (-6, 4). This line is the diameter of a circle. What is its area?
 10. Graph the following points, given the values of x and y .

(a)	x	-8	-3	-1	4	6
	y	-5	0	2	7	9
(b)	x	-10	-7	0	3	10
	y	10	8	3	1	-3

} Join the points obtained in each case.

11. If the graphs of sum 9 are drawn on one piece of squared paper, using one set of axes, they intersect, state the ordinates of the point of their intersection.
12. A charge of + 7 units of electricity are put upon an insulated sphere and then - 3 units are added. What is the resulting charge upon the sphere?

ADDITION AND SUBTRACTION

It has been already shown that to combine several plus or several minus quantities *addition* must be done, while to combine a plus with a minus quantity *subtraction* must be resorted to.

Now consider the following points in connection with these two operations:—

1. Let $a = 4$. Then $3a$, i.e. $3 \times a$, $= 12$, $4a = 16$, and $5a = 20$.

It follows that—

(a) $+3a + 4a = +7a$ (i.e. *three* fours and *four* fours make *seven* fours); and

(b) $+3a - 4a = -a$ (or $-1a$. Note, the 1 is never stated).

(c) $+3a + 4a - 5a = 7a - 5a = 2a$.

The substitution of 4 for a in each of these examples will prove their accuracy.

2. Add together $+6a^2b + 3a^2b - 7a^2b + 4a^2b - 3a^2b$.

This is like adding $6s + 3s - 7s + 4s - 3s$, where s stands for shillings.

$$\begin{array}{r} +6a^2b + 3a^2b + 4a^2b = +13a^2b \\ -7a^2b - 3a^2b \quad \quad = -10a^2b \end{array} \quad = +3a^2b.$$

You will thus note that the result of *adding* $+13a^2b$ and $-10a^2b$ is a subtraction sum.

3. Add together $a^2 + bc$, $4a^2 - 3bc$, $-2a^2 + 4bc$, $a^2 - 3bc$.

This is done by setting the terms having a^2 and the terms with bc under one another just as in setting down items of money as pounds and shillings in separate columns.

In the first column—

$$\begin{array}{r} a^2 + bc \\ 4a^2 - 3bc \\ -2a^2 + 4bc \\ a^2 - 3bc \\ \hline 4a^2 - bc \end{array} \quad \begin{array}{l} +a^2 + 4a^2 - 2a^2 + a^2 = +4a^2. \\ \\ \\ \end{array}$$

In the second column—

$$+bc - 3bc + 4bc - 3bc = -bc.$$

Thus the answer is $4a^2 - bc$.

4. Some of the difficulties of subtraction are cleared up by considering the following:—

+5s.	subtracted from	+10s.	leaves	+5s.	(1).
+4s.	”	”	”	+6s.	(2).
+2s.	”	”	”	+8s.	(3).
+1s.	”	”	”	+9s.	(4).
0s.	”	”	”	10s.	(5).
-1s.	”	”	”	11s.	(6).
-2s.	”	”	”	12s.	(7).
-4s.	”	”	”	14s.	(8).

Items 1, 2, 3, and 4 show that the operation of subtracting a plus quantity from a greater plus quantity is the same as changing the sign of the subtrahend and adding. Thus $+10s.$ less $+4s. = +10s. - 4s. = 6s.$

Items 6, 7, and 8 show that the operation of subtracting a minus quantity from a plus quantity results in *adding* the terms together. It is difficult to realize at first that -6 subtracted from $+10$ equals $+16$, but the above will do much to show the truth of this statement.

The difficulty can always be overcome by the following rule:—To subtract one quantity from another, change the sign of the former, and proceed as in addition.

$$\begin{array}{lll} \text{Thus } +8a \text{ subtracted from } +12a = +12a - 8a = +4a. \\ +8a & \text{,,} & -12a = -12a - 8a = -20a. \\ -8a & \text{,,} & +12a = +12a + 8a = +20a. \\ -8a & \text{,,} & -12a = -12a + 8a = -4a. \end{array}$$

Example.—Subtract $6a^2 - 3a^2b + 4c^2$ from $5a^2 + 7a^2b + 8c^2$. Arrange the terms with like letters and powers under one another as before.

$$\begin{array}{r} \text{Minuend} \quad 5a^2 + 7a^2b + 8c^2 \\ \text{Subtrahend} \quad 6a^2 - 3a^2b + 4c^2 \\ \hline \text{Difference} \quad -a^2 + 10a^2b + 4c^2 \end{array}$$

Steps

First term—

$$+5a^2 - 6a^2 \text{ (changing sign) } = -a^2 \text{ (adding).}$$

Second term—

$$+7a^2b + 3a^2b \text{ (changing sign) } = 10a^2b \text{ (adding).}$$

Third term—

$$+8c^2 - 4c^2 \text{ (changing sign) } = +4c^2 \text{ (adding).}$$

Thus result is $-a^2 + 10a^2b + 4c^2$.

5. Substitution—

Example.—If $a = 2.5$, $b = 4.7$, $c = 1$, find the value of $a^2 + 8b - 4c$.

$$\begin{aligned} a^2 &= 6.25, +8b = +37.6, -4c = -4. \\ \therefore a^2 + 8b - 4c &= 6.25 + 37.6 - 4 = 43.85 - 4 \\ &= 39.85 \end{aligned}$$

Exercises 53

1. Add together—

(a) $+3a, +5a, -6a, -3a, +7a$.

(b) $5l + 6l - 2l + 18l$.

(c) $3m + 4m - 12m + 15m - 6m$.

2. Add together—

(a) $2a + 3b, 4a - 5b, b - 3a, 6b - 4a, 8a - 2b$.

(b) $3a^2 - 4ab, 2a^2 + 7ab, 5a^2 + 6ab, -4a^2 - 18ab$.

(c) $a + b + c, 3a + 4b - 5c, 5a - 6b + 2c, -6a + 3b + c$.

3. If $a = 1$, $b = 2$, and $c = 1.5$, find the numerical values of the results of Question 2.4. Add together $6a - 12b + 4c, 8a - 7b + 3c, 8b + 3c + a, -6c - 4a$.5. If $a = 10$, $b = 3$, $c = 5$, find the value of the result of Question 4.6. Add together $x^3 - 6x^2y + 4xy^2 - 15y^3, 3x^3 + 17x^2y - 12xy^2 + 14y^3, 14xy^2 - 16x^2y$, and $15y^3 - 12xy^2 + 9x^2y - 3x^3$. What is the numerical value of the result if $x = 5$ and $y = 2$?7. From $+16a$ take $+7a, -3a, +18a, +16a$ in turn. If $a = 1$ ft., give the results in each case in inches.8. From $3a^2 - 4ab$ take $2a^2 + 5ab$.9. From $a^2 - 3b + 4c$ take $a^2 - 4b - 7c$.10. Subtract $4yz - 3xy - 2yx$ from $4yz + 3xy - 2yx$.

11. What must be taken from—

(a) 16 to leave 10?

(b) $+13$ to leave -1 ?

(c) $+16$ to leave -8 ?

(d) -12 to leave $+6$?

(e) -12 to leave -6 ?

(f) $a + b$ to leave $+a$?

(g) a to leave $a - c$?

(h) $a + b + c$ to leave $a + c$?

(k) $6a + 7b - 3c$ to leave $-6a$?

(m) $2a + 3b$ to leave $+6c$? [$+4b$?

12. What must be added to—

(a) 6 to make -8 ?

(b) -4 to make $+10$?

(c) a to make $2a + 4c$?

(d) $a + 3b$ to make $4a - 2b$?

(e) $3a - 2b + c$ to make c ?

(f) $4a + b$ to make $-5a + c$?

13. When $a = 4$, $b = 3$, $c = 5$, and $d = 0$, find the values of—

(a) $a^2 + 2ab + b^2$.

(b) $a^3 - 3a^2b + 3ab^2 - b^3$.

(c) ab^2c^3 .

(d) ab^2c^3d .

(e) $\sqrt{a} + 6c$.

(f) $+\frac{a}{b+c} - \frac{b}{3c-b} + \frac{c}{3c^2+2bc-b^2}$.

(g) $\sqrt{4a^2 - 20ab + 25b^2}$.

Brackets.—When terms are contained within a pair of brackets they must be considered as *one* quantity, and these may be either plus or minus, just as single terms are plus or minus. Thus $+(6 + 5)$ is the same as $+11$ and $-(7 + 6)$ is the same as -13 . Similarly—

- $(a) \quad +(10 - 5) = +5 \text{ and } -(10 - 5) = -5.$
 $(b) \quad +(10 - 7) = +3 \quad ,, \quad -(10 - 7) = -3.$
 $(c) \quad +(10 - 9) = +1 \quad ,, \quad -(10 - 9) = -1.$
 $(d) \quad +(10 + 10) = 0 \quad ,, \quad -(10 - 10) = 0.$
 $(e) \quad +(10 - 11) = -1 \quad ,, \quad -(10 - 11) = +1.$
 $(f) \quad +(10 - 13) = -3 \quad ,, \quad -(10 - 13) = +3.$

Note how the increasing subtrahend within the brackets (it increases from 5 to 13) gradually converts the value of the terms within the brackets from a plus to a minus quantity. On examining the result of putting a plus or minus sign before the brackets, in each case it will be seen that a plus before a plus quantity in brackets, or a minus before a minus quantity in brackets, results in a *plus* answer, while a plus sign before a minus quantity in brackets, or vice versa, results in a *minus* answer.....1.

Thus $+(+6) = +6$, $(-6) = -6$, $- (+6) = -6$, and $- (-6) = +6$.

Examine also the following:—

- $(a) \quad 10 + (6 - 4) = 10 + 2$, or $10 + 6 - 4$, i.e. 12.
 $(b) \quad 10 + (6 - 8) = 10 + (-2)$, i.e. $10 - 2$, or $10 + 6 - 8 = 8$.

Note, when $+$ comes before the brackets they may be removed.....2.

Again—

- $(c) \quad 10 - (6 - 4) = 10 - (+2)$, i.e. $10 - 2$, or $10 - 6 + 4 = 8$.
 $(d) \quad 10 - (6 - 8) = 10 - (-2)$, i.e. $10 + 2$, or $10 - 6 + 8 = 12$.

Note, when $-$ comes before the brackets, to arrive at the right result the signs of the terms within the brackets must be changed when the brackets are removed.....3.

You can now understand these **Rules**.

Rule 1. When removing a pair of brackets, if a minus precede it, change the signs of the terms within. No change is needed if plus precedes it2 and 3.

Rule 2. Like signs give plus, unlike signs give minus1.

“Like” signs are two plus or two minus signs, e.g. $+(+3) = +3$, $-(-3) = +3$.

“Unlike” signs are a plus sign with a minus sign, e.g. $+(-3) = -3$, $-(+3) = -3$.

The following examples will show the use and manipulation of brackets:—

Example 1.—One form of brackets used.

Simplify—

$$3x - 4y - (3x + 2y - 3z) - (4x - 2y + z) + (x - y + z).$$

To remove the brackets change the signs of the terms within if a minus precedes. The expression becomes—

$$3x - 4y - 3x - 2y + 3z - 4x + 2y - z + x - y + z.$$

Collecting the terms—

$$\begin{array}{rcl} 3x - 3x - 4x + x & = & -3x \\ -4y - 2y + 2y - y & = & -5y \\ +3z - z + z & = & +3z. \end{array}$$

The result is therefore $-3x - 5y + 3z$.

Example 2.—Two or three forms of brackets and a “vinculum” used.

$$10a + b - [6a - \{5b - (8a + 9b - \overline{5a + 4c})\}].$$

Note, you have three kinds of brackets: $[]$, $\{ \}$, and $()$, and a “vinculum”, which acts just like a pair of brackets.

Method.—Remove the vinculum first, and then each pair of brackets in turn, always observing the rules of signs.

Thus—

$$\begin{aligned} & 10a + b + [6a - \{5b - (8a + 9b - \overline{5a + 4c})\}] \\ &= 10a + b + [6a - \{5b - (8a + 9b - 5a - 4c)\}], \\ & \quad \text{removing vinculum} \\ &= 10a + b + [6a - \{5b - 8a - 9b + 5a + 4c\}], \\ & \quad \text{removing } () \\ &= 10a + b + [6a - 5b + 8a + 9b - 5a - 4c], \\ & \quad \text{removing } \{ \} \\ &= 10a + b + 6a - 5b + 8a + 9b - 5a - 4c, \\ & \quad \text{removing } [] \\ &= 19a + 5b - 4c, \text{ collecting the terms.} \end{aligned}$$

Exercises 54

1. What is the value of—

(a) $6a - (3a + 2a)$; (b) $10a - (5a - 4a)$; (c) $6a + (7a - 8a)$?

2. Find the value of—

(a) $63 - (9 - 15 + 24)$; (b) $2\frac{7}{8} - (1\frac{3}{8} - \frac{3}{8} + \frac{1}{4})$; (c) $a - (4a - 5b + 3c)$.

3. Simplify—

(a) $2.9a^2 - 3.75ab + (4.6a^2 + 7.35ab) - (6.72a^2 - 2.685ab) + 14.3ab$.

(b) $(a + b + c) - (a + b - c) - (a - b + c) + (b + c - a)$.

4. Simplify—

$6x + y - [3x + 2y - \{7x - 2y - (6y - 16z)\} + 4x]$.

5. Remove the brackets from the expression—

$(x + 2y) - [4x - \{3x - (2y - 5x)\}]$.

6. If
- $x = 5$
- ,
- $y = 2$
- ,
- $z = 1$
- , find the values of—

(a) $\frac{(x - y)^2}{x + y} + \frac{(y - z)^2}{y + z} - \frac{(x - z)^2}{z + x}$.

(b) $5x^2 + 4y^2 + 3z^2 - (10xy + 14yz + 16xz)$.

(c) $6xy - [(3xz + 4yz - 7xy) - 4z^2 + 7x]$.

7. When
- $x = 3$
- and
- $y = 2$
- , find the value of—

$6x - (16x^2 - 16y^2 - 2x) + 16x^2 - (16y^2 + 4y)$.

Exercises 55.—Some Revision Exercises
involving Addition and Subtraction

1. A hexagon of 2 in. side is circumscribed by a circle. Find the areas of the circle and of the hexagon by making suitable measurements. Hence deduce the area of each of the segments of the circle cut off by the sides of the hexagon.
2. The end of a house 26 ft. broad has the ridge of the roof 60 ft. and the eaves 51 ft. from the ground. Its shape is that of an isosceles triangle upon a rectangle. Find the area of the end of the house.
3. Determine the areas of each of two concentric circles whose radii are respectively 4.6 in. and 5.4 in. What is the area of the "annulus" or ring enclosed by their circumferences?
4. A rectangular piece of metal is 6 in. by 4.5 in. and is 2 in. thick. It has three circular holes 1.5 in. in diameter punched out of it. Determine the volume of metal removed from the plate, the volume of metal remaining, and its weight if 1 c. in. of the metal weighs .026 lb.
5. A lawn 16 ft. by 8.6 ft. has three circular flower beds cut out of it each 4 ft. in diameter. Find the area of turf remaining.
6. A window with a semicircular top is 18 ft. high in the centre and 5 ft. wide. What area of glass is required to glaze it?

THE EQUATION $a + b = c + d$

Equations of the form $ab = cd$ and $\frac{a}{b} = \frac{c}{d}$ have been considered in Sections II and III, and it was found that to transfer a multiplier across the equal signs to the other side of the equation it was necessary to convert it into a divisor, and vice versa.

Now consider $a + b = c + d$.

Let $a + b = 8$, then $c + d = 8$, and $a + b = c + d$ could be written thus: $5 + 3 = 6 + 2$, by giving these values to the four terms.

If $5 + 3 = 6 + 2$, then taking 5 from each side, since equals taken from equals leave equals, $5 + 3 - 5 = 6 + 2 - 5$, or $3 = 6 + 2 - 5$. The 5 disappears from one side and reappears on the other side of the equation with the opposite sign.

Similarly, as—

$$5 + 3 = 6 + 2,$$

then, adding -2 to both sides,

$$\begin{aligned} 5 + 3 - 2 &= 6 + 2 - 2, \\ \text{or } 5 + 3 - 2 &= 6. \end{aligned}$$

I.e. The 2 has been transferred, but the sign has been altered.

The same holds good if a minus quantity is transferred. It becomes plus. The transference of these terms has been effected by adding to each side of the equation terms equal to them in size but opposite in sign. Then cancelling makes it appear that the terms have been transferred to the other side of the equals sign and their signs changed.

Thus, just as a multiplier becomes a divisor, if transferred, so a plus quantity becomes a minus quantity, if transferred.

Example 1.—Solve the equation $3x - 4 = 24 - x$.

Method.—Transfer the terms containing x to the left side and the other terms to the right side of the equals sign, changing the sign when transference is made.

$$\begin{aligned} 3x - 4 &= 24 - x \\ 3x + x &= 24 + 4 \quad (-x \text{ becomes } +x, \text{ and } -4, +4) \\ 4x &= 28 \\ x &= 7. \end{aligned}$$

Example 2.—The length of a field is 3 yd. longer than its breadth. Its perimeter is 1000 yd. Find its length and breadth.

If x = its breadth in yards, then $(x + 3)$ = its length in yards. Its perimeter or distance round equals the sum of twice its length and breadth, or $(x + 3) + (x + 3) + x + x$, which equals 1000 yd.

$$\begin{aligned}\text{Thus } (x + 3) + (x + 3) + x + x &= 1000 \\ 4x &= 1000 - 6 \\ 4x &= 994 \\ x &= 248\frac{1}{2} \text{ yd.}\end{aligned}$$

The field is $251\frac{1}{2}$ yd. long and $248\frac{1}{2}$ yd. broad.

Exercises 56

1. Solve the equations—

- (a) $18x + 13 = 59 - 5x$.
- (b) $2.5x + 16 = 7.25 - 5.5x + 0.25$.
- (c) $3.5x + 3 = 2.5x + 13$.
- (d) $.003x + .0315 = .00125x + .0175x$.
- (e) $\frac{3}{8}x + \frac{5}{4}x - 2\frac{1}{4} = \frac{7}{8}x - \frac{7}{4}$.
- (f) $16 - 3x + 5x - 20 = 1\frac{1}{2}x + 7x - 17$.

2. One number is 49 greater than another number. If x is one number algebraically represent the other. The sum of these numbers is 53. Find them.

3. $.15x + 1.575 = .0625x + .875x$. Find x .

4. Solve the equations—

- (a) $8x - 4 = 7x - 1$.
- (b) $4x + 7 = 8x - 7$.
- (c) $10x - 2x + 8 = 3x + 8x + 4.25$.

5. Express algebraically (a) the sum of a , b , and c ; (b) the difference between c and the sum of a and b .

6. Express algebraically—

- (a) The cube root of the square of a multiplied by b .
- (b) The square of a added to the cube root of b .
- (c) The sum of the squares of a and b .
- (d) The square root of the sum of $+a$, $-b$, and $-c$.

7. Divide 21 in. into two parts so that one part is 9 in. longer than the other.

8. x is a number. Its half added to its third part make 15. Find x .

9. x is a number. What is the next consecutive number and the number one smaller than x ? If the sum of these three numbers is 63, what is x ? Work your sum by an equation.

10. The length of a field is three times its breadth. If x is its breadth
(c 200)

- what is its length? The perimeter of (i.e. the distance round) the field is 2400 yd., find its length, breadth, and area.
11. Three times a certain number diminished by six are equal to twice the number increased by four. State the equation and find the number.
 12. Eight times the distance between two villages plus 4 m. is 100 m. Find the distance apart of the villages.

THE SIDES OF A RIGHT-ANGLED TRIANGLE

In Section IV the ratio of two sides of the right-angled triangle was shown to be a means of measuring the angles of the triangle, and tables of sines, cosines, and tangents were considered. It is now necessary to consider the three sides apart from the angles, and to be able to find any side, given the lengths of the other two. The following exercise will show how this can be done.

Exercises 57

1. Construct a right-angled triangle ABC, right-angled at A. Make the perpendicular and base AB and AC 3 in. and 4 in. long respectively. Carefully measure the hypotenuse BC. On each of these sides construct squares, and calculate their areas. Add the areas of the squares on the perpendicular and base. How does your result compare with the area of the square on the hypotenuse?
2. Repeat Question 1, making the perpendicular and base 12 cm. and 5 cm. respectively. Add the areas of the squares on these sides and compare your answer with the area of the square on the hypotenuse which you measure.
3. Repeat 1 and 2 with any right-angled triangle, carefully measuring the sides to the nearest millimetre. Does the same general result hold true?
4. ABCD is a square. By bisecting its sides and joining the points of bisection to the angular points in order fig. 42 is produced. Do this to as large a scale as possible, and carefully cut out the triangles AEB, BFC, DGC, AHD, and the square EFGH. The four triangles are equal, and will make a square if put side by side. Thus fig. 42 has been converted into two squares. Its area is thus equal to the sum of the areas of the two squares. Finally these three squares will be found to be the squares on the three sides of any one of the four equal right-angled triangles. Try it.

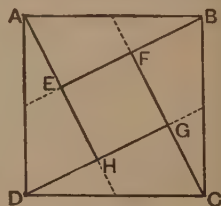


Fig. 42

5. State what you have discovered by working the above four exercises, and show that, if a is the length of the hypotenuse of a right-angled triangle, and b and c its other two sides, this discovery may be represented by the equation $a^2 = b^2 + c^2$.
6. Show by Fig. 42 (a) that it is possible to divide a square into 5 equal squares; (b) that the triangle AEB equal to the square EFGH is double its altitude.

The square on the hypotenuse (h) of a right-angled triangle is equal in area to the sum of the areas of the squares on its perpendicular (p) and base (b).

$$\text{Or } h^2 = p^2 + b^2. \quad \therefore h = \sqrt{p^2 + b^2}.$$

By transferring p^2 to the other side of the equation we get—

$$h^2 - p^2 = b^2, \text{ or } b^2 = h^2 - p^2.$$

$$\therefore b = \sqrt{h^2 - p^2}.$$

$$\text{Similarly, } h^2 - b^2 = p^2, \text{ or } p^2 = h^2 - b^2.$$

$$\therefore p = \sqrt{h^2 - b^2}.$$

Example.—The slant height of a cone is 7 in., the diameter of its base is 5 in. Find its vertical height.

In fig. 43, h is 7 in., b is 2.5 in. (radius), and p is x in. where x is the vertical height required.

$$\text{Now } p^2 + b^2 = h^2.$$

$$\therefore x^2 + 2.5^2 = 7^2.$$

$$\text{i.e. } x^2 = 7^2 - 2.5^2 = 49 - 6.25 = 42.75.$$

$$\therefore x = \sqrt{42.75} = 6.53 \text{ in.}$$

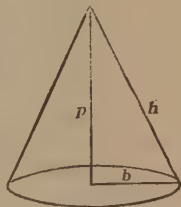


Fig. 43

(For further simplification of work see Application of Factors in the next section.)

Exercises 58

1. The hypotenuse of a right-angled triangle is 6.5 in., and one side is 4.3 in., find the length of the other side.
2. Find the length of the hypotenuse of a right-angled triangle whose sides are 6.38 m. and 4.35 m. long.
3. A ladder 40 ft. long, tilted against a perpendicular wall with its foot 5 ft. from the wall, just reaches a window sill. How high is the sill from the ground?

4. The vertical height of a cone is 14.6 in., and the diameter of the base is 7 in. Find its slant height.
5. The length of one side of the square base of the Great Pyramid is 732 ft., and its height is 480 ft. How many feet will a man traverse in climbing to the top by the shortest course?
6. By using this slant height of the Great Pyramid, and the length of the side of the square base, determine the length of one of the edges of the pyramid.
7. A reservoir is shaped like an inverted truncated square pyramid. The side of the upper square is 543 ft., and the side of the square bottom of the reservoir is 217.2 ft. A stone rolling down one side travels 271.5 ft. to reach the bottom. Find the depth of water the reservoir will hold.
8. A table is 6 ft. long by 4.3 ft. broad. Find the distance of the opposite corners from one another.
9. Two ships leave harbour at the same time. One travels north at the rate of 16 knots per hour, and the other goes east at the rate of 18 knots per hour. How far are they apart in nautical miles (knots) in 3 h.?
10. The ridge of a roof is 54 ft. from the ground, and its two eaves are 43 ft. from the ground. If the house covers a rectangular base 60 ft. by 27 ft. determine the area of tiling required for the roof, and the number of tiles required if 11 tiles cover 3 sq. ft.
11. A ladder 47 ft. long reaches the top of a wall 42 ft. high. How far is the foot of the ladder from the wall?
12. Draw a tangent to a circle of 3.5 cm. radius from a point 7.5 cm. from its centre (Section IV). Determine the length of the tangent.
13. BAC is an angle bisected by the line AD, and at D a circle is drawn touching the arms of the angle. If AD is 5 in., and the radius of the circle is 1.5 in., determine the lengths of the arms AB and AC between A and their points of contact with the circle.
14. A tangent 13.6 cm. has to be drawn from a point A to a circle whose centre B is 17 cm. from A. Find the radius of the circle.

SUMMARY

The **signs**, **Plus** and **Minus**, are placed before magnitudes to denote their *quality*, those which are plus being opposite in quality to those which are minus. It is necessary to dissociate the idea of addition and subtraction from these signs to obtain a correct idea of their significance.

Addition and **Subtraction** are the operations performed to obtain in simplest form the result of combining together *terms* of like or different sign. Thus if terms of like sign are com-

bined, they are added, but if terms of unlike sign are combined, they are subtracted.

Method of Subtraction.—To subtract one expression (subtrahend) from another (the minuend), change the signs of the subtrahend, and combine as in addition.

Brackets are used to express the idea that the expression within them must be treated as one term.

Equations.—*Rules.*

1. A multiplier transferred across the sign of equality becomes a divisor.

2. A divisor transferred across the sign of equality becomes a multiplier.

3. A plus quantity transferred across the sign of equality becomes a minus quantity.

4. A minus quantity transferred across the sign of equality becomes a plus quantity.

The Right-angled Triangle.—If a is the hypotenuse of a right-angled triangle, and b and c are the remaining sides, then the relationship existing among the lengths of the sides is shown by the equation—

$$a^2 = b^2 + c^2.$$

SECTION VI.—PRODUCTS, QUOTIENTS, AND FACTORS

Contracted multiplication and division have been dealt with in Section I, while in Sections II to IV products and quotients have been considered without regard to the quality of the magnitudes taken. The signs “plus and minus” must now be taken in connection with these operations.

Exercises 59.—Some Revision Exercises

1. 305.4×16.37 (use contracted method of multiplication).
2. Find the area of a field 205.7 Hm. long by 158.9 Hm. broad. Express your result in square kilometres.
3. $43.058 \div 6.387$ (contracted method).
4. Find the diameter of a circle whose circumference is 9068.3 m. long.

5. $\frac{3.65}{x} = \frac{14.89}{17.65}$ Find x .
6. The base of a cylinder is 57.83 sq. cm. and its height is 19.87 cm. Determine its volume and its weight if 1 c.cm. of the material of which it is composed weighs 39.6 g.
7. The base of a triangle of 316 sq. in. area is 18.7 in. Find its height.
8. The area of a triangle is x sq. in., and its height is m in. Find the length of its base.
9. Write down algebraically—
 (a) m times the fifth power of x . From this take n times the square of y . Multiply your result by $6a$.
 (b) Divide the square of the sum of a and b by x times the square of y .
10. Find the area of a parallelogram whose sides measure 16.85 and 7.39 cm. respectively, and include an angle of 70° .

Make carefully the following observations:—

A.—26 means (2 tens + 6 units) or (20 + 6).

xy means $x \times y$. The x has no place value, neither has the y .

x^3 means $x \times x \times x$.

Thus the product of 26, a , xx , yyy is $26 \times a \times x^2 \times y^3 = 26ax^2y^3$, and this is the simplest way in which it can be written down. The order of the letters does not matter $26ax^2y^3 = 26y^3x^2a = 26x^2ay^3$, &c.

B.—Multiply 2^3 by 2^4 and x^3 by x^4 .

$$\begin{aligned} \text{Method.}—2^3 &= 2 \times 2 \times 2 & x^3 &= x \cdot x \cdot x \\ \text{and } 2^4 &= 2 \times 2 \times 2 \times 2 & \text{and } x^4 &= x \cdot x \cdot x \cdot x \\ \therefore 2^3 \times 2^4 &= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^7 \\ &\therefore x^3 \times x^4 &= x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x &= x^7 \\ \text{i.e. } 2^3 \times 2^4 &= 2^{3+4} = 2^7 & \therefore x^3 \times x^4 &= x^{3+4} = x^7. \end{aligned}$$

This shows that in *multiplying* together the *powers* of a number the *indices* are *added*, e.g. $y^3 \times y^5 = y^8$; $x^{1.5} \times x^{3.2} = x^{4.7}$; $10^2 \times 10^{4.5} = 10^{6.5}$.

C.—Divide 2^3 by 2^2 and x^4 by x^2 .

$$\begin{aligned} 2^3 \div 2^2 &= \frac{2^3}{2^2} = \frac{\cancel{2} \times \cancel{2} \times 2}{\cancel{2} \times \cancel{2}} = 2, \text{ i.e. } 2^3 \div 2^2 = 2^{3-2} = 2. \\ x^4 \div x^2 &= \frac{x^4}{x^2} = \frac{\cancel{x} \cdot \cancel{x} \cdot x \cdot x}{\cancel{x} \cdot \cancel{x}} = x^2, \text{ i.e. } x^4 \div x^2 = x^{4-2} = x^2. \end{aligned}$$

Thus in obtaining the *quotient* of the *powers* of a number the *indices are subtracted*, e.g. $x^8 \div x^3 = x^5$; $14x^3y^2 \div 2x^2y^2 = 7x$; $x^5 \div x^4 = x$.

D.—The law of signs.

Let $OA = +7$, then $OA_1 = -7$, $OB = +21$, and $OB_1 = -21$.

$+7$ (i.e. OA)	expressed	$+3$ times	results in	OB or $+21$.
-7 (i.e. OA_1)	"	$+3$ " "	"	OB_1 " -21 .
$+7$ (i.e. OA)	"	-1 times	must result in	OA_1 , i.e. -7 .
$\therefore +7$ (i.e. OA)	"	-3 " "	"	OB_1 " -21 .
and -7 (i.e. OA_1)	"	-1 " "	"	OA " $+7$.
$\therefore -7$ (i.e. OA_1)	"	-3 " "	"	OB " $+21$.

These statements will become clear if worked out in fig. 44, if it is remembered from Section V that the *directions* in which

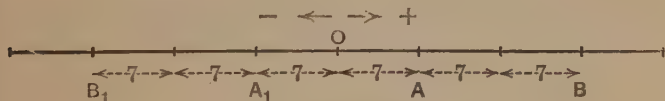


Fig. 44

these sevens are marked are either *positive* (to the right) or *negative* (to the left), and that a *movement, negative to a negative direction, must take place in an opposite direction to it, i.e. must result in a positive measure.*

Thus $(+7) \times (+3)$ or $(-7) \times (-3)$ give $+21$ as product;
and $(+7) \times (-3)$ or $(-7) \times (+3)$ give -21 as product.

I.e. Like signs (two plus or two minus signs) give plus; unlike signs (a plus and a minus, or vice versa) give minus.

Note.—This law has been arrived at already in Section V (brackets).

Examples.—1. $+6abc \times +4cd = +24abc^2d$.

2. $+3ax \times -2a^2y = -6a^3xy$.

3. $-2x^2 \times -4x^2 = 8x^4$.

Exercises 60

1. If $x = 13.6$ find the value of x^2 , x^3 , and $\sqrt{x}$.

2. Write down the results of the following:—

$$(a) a^3 \times a^7 =$$

$$(b) x^{12} \times x^3 =$$

$$(c) y^{\frac{1}{2}} \times y^{\frac{1}{3}} =$$

$$(d) m^{\cdot 5} \times m^{\cdot 4} =$$

$$(e) l^{1.6} \times l^{1.5} =$$

$$(f) r^{\frac{2}{3}} \times r^{1\frac{1}{3}} =$$

$$(g) 7^3 \times 7^2 =$$

$$(h) 10^{1.328} \times 10^{.30103} =$$

3. Give the answers to the following:—

$$(a) x^{10} \div x^7 =$$

$$(b) a^5 \div a^{2.5} =$$

$$(c) a^{\frac{3}{4}} \div a^{\frac{1}{2}} =$$

$$(d) s^{16} \div s^4 =$$

$$(e) 10^{1.4771} \div 10^{.4771} =$$

$$(f) t^6 \div t^{.36} =$$

$$(g) y^{3.68} \div y^{1.27} =$$

$$(h) m^{1\frac{1}{2}} \div m^{\frac{1}{2}} =$$

4. Complete the statements:—

$$(a) 6a^2bc \times 4ab^3 =$$

$$(b) 4c^2x \times 12yz =$$

$$(c) m^2a^3xy \times 16ax^2 =$$

$$(d) l^2st^3 \times 3as =$$

$$(e) 7a^3bc^2 \div a^2c.$$

$$(f) 21x^4b^4 \div 3x^2b.$$

$$(g) 17ab^2c \div 2cab^2.$$

$$(h) x^8y^{16} \div 8x^4y^{12}.$$

5. $(+17a^3b^2c) \times (+16abc^3)$.

6. $(+48x^3y^2) \div (-24xy)$.

7. Multiply $-3.6x^2$ by $-4.2abx$.

8. Multiply $-6m^2n$ by $+8mx^2y$, and divide your result by $+12m^3xy$.

9. If $3x^2y = 7\frac{2}{3}$, find the value of x^2y ; and, given that $x = \frac{2}{3}$, find from your result the value of y .

10. The volume of a cone is $\frac{1}{3}\pi r^2h$ where r is the radius of the circular base and h is the perpendicular height. Find h when r is $7\frac{1}{2}$ in. and the volume is 207.9 c.in. (Take π as $\frac{22}{7}$.)

MULTIPLICATION OF ALGEBRA

Example 1.—It is required to multiply $7x^2 + 3x^3y^2 - 4xy^3$ by $3xy$.

Method—

$$7x^2 + 3x^3y^2 - 4xy^3$$

$$\underline{3xy}$$

$$\underline{21x^3y + 9x^4y^3 - 12x^2y^4}$$

Analysis of method—

$$+7x^2 \times +3xy = +21x^3y.$$

$$+3x^3y^2 \times +3xy = +9x^4y^3.$$

$$-4xy^3 \times +3xy = -12x^2y^4.$$

Order of work in determining each term of product—

1. Determine the sign. (Like signs give plus; unlike, minus.)
2. Find product of numbers or *numerical coefficients*.
3. Set down all the letters, adding the indices where necessary.

Example 2.—It is required to multiply $7x^2 + 8xy - 9y^2$ by $4x^2 - 2xy - y^2$. Take each term in the multiplier and proceed as in Example 1. Arrange like terms under one another, and add. Compare with 6835×365 .

$$\begin{array}{rcl}
 7x^2 + 8xy - 9y^2 & & \\
 4x^2 - 2xy - y^2 & & \\
 \hline
 28x^4 + 32x^3y - 36x^2y^2 & = & 4x^2 \text{ times} \\
 -14x^3y - 16x^2y^2 + 18xy^3 & = & -2xy \text{ ,,} \\
 -7x^2y^2 - 56xy^3 + 9y^4 & = & -y^2 \text{ ,,} \\
 \hline
 28x^4 - 18x^3y - 59x^2y^2 - 38xy^3 + 9y^4 & = & (4x^2 - 2xy - y^2) \text{ times.} \\
 \hline
 \end{array}$$

Compare—

$$\begin{array}{rcl}
 6835 & & \\
 365 & & \\
 \hline
 2650500 & = & 300 \text{ times} \\
 410100 & = & 60 \text{ ,,} \\
 34175 & = & 5 \text{ ,,} \\
 \hline
 3,094,775 & = & 365 \text{ ,,} \\
 \hline
 \end{array}$$

Analysis of method—

$$\begin{array}{rcl}
 (7x^2 + 8xy - 9y^2) \times 4x^2 & = & 28x^4 + 32x^3y - 36x^2y^2 \text{ (cf. Ex. 1).} \\
 \text{,,} \quad \text{,,} \quad \times -2xy & = & -14x^3y - 16x^2y^2 + 18xy^3 \text{ ,,} \\
 \text{,,} \quad \text{,,} \quad \times -y^2 & = & -7x^2y^2 - 56xy^3 + 9y^4 \text{ ,,}
 \end{array}$$

Note 1.—To arrange like terms under one another, the terms of each part product have to be removed one place to the right.

2. In a term, say $-36x^2y$, which means $-(36 \times x \times x \times y)$, the number and each letter is a *factor* of the expression, just as 3 and 4 are factors of 12. 36 is called the *numerical coefficient* and x^2y the *literal coefficient* of the term.

3. In Example 2 the literal coefficients have been arranged in a certain order. The powers of x decrease while the powers of

y increase. Thus $7x^4 - 2 + 3x - 4x^3 + 2x^2$ is an expression not arranged in order. Arranged in accordance with the "descending powers" of x it will be—

$$7x^4 - 4x^3 + 2x^2 + 3x - 2,$$

and arranged according to the "ascending powers" of x it will be—

$$-2 + 3x + 2x^2 - 4x^3 + 7x^4.$$

Before multiplying and dividing, arrange the terms of your expression in order. This saves difficulty in the after arrangement of the sum.

Exercises 61

1. Arrange according to the descending powers of x and ascending powers of m —

$$(a) 7xy^6 - 6y^7 - 3x^7 - 4x^6y + 12x^2y^5 + 6x^3y^4 - 8x^5y^2.$$

$$(b) m^5 - 4m - 6 + 2m^4 - m^6 + 7m^3 - 8m^2.$$

2. Multiply $3x^2 - 4xy + 6y^2$ by each of the following in turn—

$$(a) + 6; (b) 3x; (c) - 3x; (d) + 4y; (e) - 7y; (f) + 3a;$$

$$(g) - 7x^2y^2; (h) - 8z.$$

3. Find the product of $7p^4 - 3p^3 + 2p^2 - p + 8$ and $- 6p^2$.
4. The sides of a rectangle measure $8m + 4d$ and $6m$ respectively. Find its area. If $m = 10$ dm. and $d = 1$ dm., show that your answer will give the area in square decimetres.
5. Multiply $a^3 - a + 3a^2 + 7$ by $a - 1$.
6. Find the products of—

$$(a) 7x^4 + 2x^3 + 3x^2 - 4 \text{ and } -x + 2.$$

$$(b) a^2b - 3ab^2 + b^3 \text{ and } a + b.$$

$$(c) a^2 + 2ab + b^2 \text{ and } a - b.$$

7. Simplify: $(a + b + c)(a - b - c)$. *Note.*—Two brackets arranged without intervening sign are like two letters without intervening sign. Multiplication is inferred.
8. Simplify—
 $(a) (a + b)(a + b). \quad (d) (a + b)(a - b)(a^2 + b^2).$
 $(b) (a - b)(a - b). \quad (e) (a + b)(a^2 - ab + b^2).$
 $(c) (a - b)(a + b). \quad (f) (a - b)(a^2 + ab + b^2).$
9. $3x(4a - 3b + 2c) =$. *Note* that a term and a bracket without intervening sign signifies multiplication of bracket by term.
10. Simplify—
 $(a) 4y(3xy - 2y).$
 $(b) -x^2(3x^2 - 2xy + 4y^2).$

FACTORS

It is often convenient to proceed in the opposite direction to multiplication, and, given a product, to find the multiplicand and multiplier which produced it; or, in other words, to find the *factors* of the expression. This process is termed factorization. The following exercises serve to introduce some of the principles.

Note.—The factors of 12 are 4 and 3 for $4 \times 3 = 12$, and also as will be seen 2 and 6 or 2 and 2 and 3; and the factors of $6a$ are 6 and a for $6 \times a = 6a$, and also for like reason 3 and $2a$; 2 and $3a$.

Exercises 62

1. Name the factors of 49, 21, 51, 119, ab , $7bc$, $13m^2$, $14mn$.
2. Give as many groups of two factors as possible of—
(a) 128; (b) $16a^2b$; (c) $25a^3b^2$; (d) 104.
3. What two factors have been multiplied together to produce—
(a) $3a - 3b$; (b) $7m + 7n$; (c) $ax^2 - ay^2$; (d) $x^2y + x^2z$?

Note.—Arrange one factor as an expression in brackets.

4. What are the factors of 21 and 27? Name their common factor.
5. What are the factors of $3a^2b$ and $9ab^2$? Give their common factor.
6. Multiply (a) $x + 2$ by $x + 3$, (b) $x + 7$ by $x + 6$, and (c) $x + 3$ by $x + 5$, and compare your results. Each product has three terms. State—

(i.) How the third term of the answer has been obtained in each case.

(ii.) How the numerical coefficient of the second term has been obtained.

7. Multiply mentally as the result of your investigation in 6—
(a) $x + 7$ by $x + 2$ and (b) $x + 4$ by $x + 8$, and state the factors of: (c) $x^2 + 5x + 6$, and (d) $x^2 + 7x + 10$.
8. Multiply (a) $x - 2$ by $x - 3$, (b) $x - 7$ by $x - 6$, and (c) $x - 3$ by $x - 5$. Compare your results with those of Sum 6, and with one another. State—
(i.) How the third terms of your answers are obtained.
(ii.) How, in the second term, the coefficient of x is the sum of the numbers.
(iii.) Why the sign of the third term is plus and that of the second term is minus.

9. From your observations set down answers for $(x + 7)(x + 3)$, $(x - 7)(x - 3)$, $(x - 9)(x - 2)$, $(x - 4)(x - 8)$; and the factors of $x^2 - 7x + 12$ and $x^2 - 7x + 10$.
10. Write down 3^2 , 5^2 , $(3m)^2$, $(x + 2)^2$, $(x + 3)^2$, $(x - 8)^2$, $(x - 1)^2$. Try to do this without working on paper, using your foregoing observations.
11. Work fully $(x + 1)(x - 1)$, $(x + 2)(x - 2)$, $(x + 3)(x - 3)$. Can you now predict the results of $(x + 5)(x - 5)$, $(x + 6)(x - 6)$, $(x - 7)(x + 7)$?
12. Work $(x - 5)(x + 4)$, $(x - 8)(x + 3)$, $(x - 4)(x + 7)$, $(x - 2)(x + 9)$, and make observations for multiplying expressions containing numerical terms of opposite signs.

Type 1. Two Factors with Second Terms of Like Sign.

$$(a) (x + 4)(x + 7) = x^2 + 11x + 28.$$

$$(b) (x - 4)(x - 7) = x^2 - 11x + 28.$$

Note.— $+28 = (+4) \times (+7)$ as in (a), or $(-4) \times (-7)$ as in (b).

$$+11 = +7 + 4 \text{ and } -11 = -4 - 7.$$

Thus to obtain 7 and 4 for the number terms of the factors of $x^2 + 11x + 28$ there are required numbers which *multiplied* give 28 and *added* give 11.

Example.—Factorize $x^2 - 5x + 6$.

Steps—(a) $(x \quad)(x \quad)$.

(b) What numbers *multiplied* give 6 and *added* give 5? *Ans.* 3 and 2.

$$(c) (x - 3)(x - 2).$$

(d) Signs—6 is *plus*. $\therefore$ Signs are alike. $5x$ is minus. $\therefore$ Signs are both minus.

$$(e) (x - 3)(x - 2).$$

$$\text{Thus } x^2 - 5x + 6 = (x - 3)(x - 2).$$

The steps are given as a guide to the order of reasoning mentally done.

Exercises 63

Factorize—

- | | | |
|-----------------------|--------------------------|----------------------------|
| 1. $x^2 + 4x + 3$. | 5. $x^2 - 9x + 20$. | 9. $p^2q^2 - 2pq + 1$. |
| 2. $x^2 - 7x + 12$. | 6. $x^2 - 16x + 39$. | 10. $b^2c^2 - 5bc + 6$. |
| 3. $x^2 - 10x + 24$. | 7. $x^2 - 7xy + 12y^2$. | 11. $a^2 - 102x + 200$. |
| 4. $x^2 + 8x + 15$. | 8. $m^2 + 8mn + 15n^2$. | 12. $x^2 - 19xy + 60y^2$. |

Type 2. Two Factors with Second Terms of Unlike Sign.

$$(a) (x + 4)(x - 7) = x^2 - 3x - 28.$$

$$(b) (x - 4)(x + 7) = x^2 + 3x - 28.$$

Note.— $-28 =$ product of -7 and $+4$, or $+7$ and -4 . 3 of $3x$ is -3 when -7 and $+4$ are added, and $+3$ when $+7 - 3$ are added.

Rule.—When last sign of product is *plus* the signs of the factors are *like* (Type 1); when last sign is *minus* the signs of the factors are *unlike* (Type 2).

To obtain 7 and 4 for the “number terms” of the factors of $x^2 - 3x - 28$ there are required numbers which *multiplied* give -28 and *added* give -3 .

Example.—Factorize $x^2 - xy - 20y^2$.

Steps—(a) $(x \quad y)(x \quad y)$.

(b) What numbers *multiplied* give $+20$ and *added* give -1 ?

Ans. -5 and $+4$.

(c) $(x - 5y)(x + 4y)$.

$$\therefore x^2 - xy - 20y^2 = (x - 5y)(x + 4y).$$

There is, however, a longer and perhaps more simple method.

Example.—Factorize $x^2 + 3xy - 18y^2$.

Steps—(a) $(x \quad y)(x \quad y)$.

(b) What numbers *multiplied* make 18 and *subtracted* (because -18) make 3 ? *Ans.* 6 and 3 .

(c) $(x \quad 6y)(x \quad 3y)$.

(d) Signs. $-18y^2$ shows signs *unlike*. $+3xy$ shows greater number is plus. $\therefore +6$ and -3 .

(e) $(x + 6y)(x - 3y)$.

Exercises 64

Factorize—

1. $x^2 - 2x - 3$.	5. $x^2 + 5x - 6$.	9. $b^2 + 9b - 90$.
2. $x^2 - x - 30$.	6. $a^2 - 5ab - 84b^2$.	10. $p^2 + 15p - 126$.
3. $x^2 - 35x - 200$.	7. $m^2 - 3mn - 10n^2$.	11. $m^2 - 4m - 221$.
4. $x^2 + 3x - 28$.	8. $x^2y^2 + xy - 20$.	12. $x^2 - 54x - 360$.

Type 3. Factors Identical, or Differing only in Signs.

(a) $(x + 3)(x + 3) = x^2 + 6x + 9$. Cf. Type 1.

(b) $(x - 3)(x - 3) = x^2 - 6x + 9$. „

(c) $(x - 3)(x + 3) = x^2 - 9$. Cf. Type 2. The x term disappears. No difficulty will be experienced with (a) and (b).

Example.—Factorize $25x^2 - 36$.

Steps—(a) $(x \quad)(x \quad)$.

(b) Signs. -36 shows signs unlike. $\therefore (x - \quad)(x + \quad)$.

(c) $\sqrt{25} = 5$, $\sqrt{36} = 6$. $\therefore (5x - 6)(5x + 6)$.

$\therefore 25x^2 - 36 = (5x - 6)(5x + 6)$.

Note.—There are no factors for $(25x^2 + 36)$.

Exercises 65

Find the value of—

$$1. (x + 8)^2.$$

$$2. (x - 5)^2.$$

$$3. (x + 2y)^2.$$

$$4. (x - 7y)^2.$$

$$5. (6 + x)^2.$$

$$6. (9 - y)^2.$$

$$7. (m + 4n)^2.$$

$$8. (p - 11q)^2.$$

Factorize—

$$9. x^2 - 25.$$

$$10. l^2 - 9t^2.$$

$$11. R^2 - r^2.$$

$$12. \pi(R^2 - r^2).$$

$$13. 16 - m^2.$$

$$14. (x + y)^2 - y^2.$$

$$15. (x + y)^2 - (x - y)^2.$$

$$16. x^4 - y^4.$$

$$17. 25x^2 - 49.$$

$$18. 625m^2 - 1.$$

$$19. 16x^4 - 81y^4.$$

$$20. 4 - (x - y)^2.$$

$$21. 6 \cdot 3^2 - 2 \cdot 7^2.$$

$$22. 146 \cdot 8^2 - 3 \cdot 2^2.$$

$$23. 1 \cdot 385^2 - 615^2.$$

$$24. (2 \cdot 6x)^2 - (5 \cdot 4y)^2.$$

Type 4. Where the Numerical Coefficient of the First Term is Greater than Unity.

$$(4x - 3)(3x + 5) = 12x^2 + 11x - 15.$$

Note.—(a) Signs follow the same rules as in Types 1 and 2.

(b) 12 in $12x^2 = 4 \times 3$.

(c) $-15 = -3 \times +5$.

(d) 11 is result of cross-multiplication of the numbers in the brackets. Thus—

$$\left\{ \begin{array}{l} +4, -3 \\ +3, +5 \end{array} \right\} (+4 \times +5) + (+3 \times -3) = +20 - 9 = +11.$$

Example 1.—Factorize $5x^2 + 22x - 48$.

Steps—(a) $(x - \quad)(x + \quad)$. Cf. Types 1 and 2.

(b) What numbers multiplied make 5? *Ans.* 5 and 1.

(c) What numbers multiplied make 48? *Ans.* 2 and 24, 4×12 , 6 and 8, and of these—

(d) What pairs of b and c cross-multiplied and the results added give $+22$?

(e) Trials (i) $\left\{ \begin{matrix} 5, -2 \\ 1, +24 \end{matrix} \right\}$ = by cross-multiplication $120 - 2 = 118$.

(ii) $\left\{ \begin{matrix} 5, -6 \\ 1, +8 \end{matrix} \right\}$ = " " $+40 - 6 = +34$.

(iii) $\left\{ \begin{matrix} 5, -8 \\ 1, +6 \end{matrix} \right\}$ = " " $+30 - 8 = +22$.

(f) $\left\{ \begin{matrix} 5, -8 \\ 1, -6 \end{matrix} \right\}$ = selected numbers. $\therefore$ factors are $\left\{ \begin{matrix} (5x - 8) \\ (x - 6) \end{matrix} \right\}$.

I.e. $5x^2 + 22x - 48 = (5x - 8)(x - 6)$.

Example 2.—Factorize $24x^2 - 64xy + 10y^2$.

Steps—

(a) $\left\{ \begin{matrix} 4, -5 \\ 6, -2 \end{matrix} \right\}$ yield -38 .

(b) $\left\{ \begin{matrix} 6, -5 \\ 4, -2 \end{matrix} \right\}$ " -32 .

(c) $\left\{ \begin{matrix} 12, -2 \\ 2, -5 \end{matrix} \right\}$ " -64 .

$\therefore$ Factors are $\left\{ \begin{matrix} 12x - 2y \\ 2x - 5y \end{matrix} \right\}$ or $(12x - 2y)(2x - 5y)$.

Exercises 66

Factorize—

$$1. 6x^2 - x - 12.$$

$$2. 6x^2 - 17x + 12.$$

$$3. 5x^2 - 38x + 48.$$

$$4. 8x^2 + 12x - 20.$$

$$5. 8x^2 + 26x + 15.$$

$$6. 14m^2 - 31mn - 10n^2.$$

$$7. 6x^2 - 19x + 15.$$

$$8. 4x^2 + 20x + 25.$$

$$9. 9x^2 + 24xy + 16y^2.$$

$$10. 4x^2 - 25x + 25.$$

$$11. 9x^2 + 12xy + 4y^2.$$

$$12. 256x^2y^2 - 160xy + 25.$$

Type 5. Where each Term has a Common Factor.

$$(a) x(a + b + c - d) = ax + bx + cx - dx.$$

$$(b) a(b + d) + c(b + d) = ab + ad + bc + cd.$$

Example 1.—Factorize $3x^2y - 6xy^2 + 9y^3$.

Steps—(a) 3 and y are factors of each of the three terms.

$$(b) 3x^2y = 3y \times x^2, -6xy^2 = 3y \times -2xy, +9y^3 = 3y \times 3y^2$$

$$(c) 3y \times (x^2 - 2xy + 3y^2) = 3x^2y - 6xy^2 + 9y^3.$$

$$\therefore 3x^2y - 6xy^2 + 9y^3 = 3y(x^2 - 2xy + 3y^2).$$

Method.—Take out the highest common factor from each term and set it before a pair of brackets containing the rest of the

expression. The work done on brackets already will show the correctness of the proceeding.

Example 2.—Factorize— $3x^3 - 6x^2 + 2x - 4$.

In this case there is no common factor to every term, but the first two terms have a common factor $3x^2$, and the last two terms have a common factor 2. Arrange these as in Example 1, and we get $3x^2(x - 2) + 2(x - 2)$.

Now, if two threes are added to four threes, the result is six threes.

Similarly, if $3x^2(x - 2)$'s are added to $2(x - 2)$'s, the result is $(3x^2 + 2)(x - 2)$'s.

Thus $(3x^2 + 2)(x - 2)$ is the final form of the expression.

$$\begin{aligned}\therefore 3x^3 - 6x^2 + 2x - 4 &= 3x^2(x - 2) + 2(x - 2) \\ &= (3x^2 + 2)(x - 2).\end{aligned}$$

Exercises 67

Factorize—

1. $4a^2b - 6ab^2$.

2. $a^3b - a^2b^2 + ab^3$.

3. $3x^3 - 6x^2$.

4. $a^2 + a^2x + abx$.

5. $96a^3b^2c - 48a^2bc^3$.

6. $20abx - 15a^2b + 25a^3b^2$.

7. $ac - ad + bc - bd$.

8. $2a^3 - 4a^2 + 6a - 12$.

9. $x^2y^2 + xyz + 2xy + 2z$.

10. $7m^3 - 3m^2 - 7m + 3$.

11. $p^3 + rp^2 - r^2p - r^3$.

12. $ac^2 + bd^2 - ad^2 - bc^2$.

General Examples of Factors of the Five given Types.—It often happens that the principles of two or even three types have to be applied to one example thus—

Factorize $6x^2 - 12xy + 6y^2 - 6a^2$.

Steps.—1. Take out common factor 6 and we get—

$$6(x^2 - 2xy + y^2 - a^2). \quad [\text{Type 5.}]$$

2. Factorize $x^2 - 2x + y^2$ and we get—

$$\begin{aligned}&6\{(x - y)(x - y) - a^2\}, \\ \text{i.e. } &6\{(x - y)^2 - a^2\}.\end{aligned}$$

3. Factorize $(x - y)^2 - a^2$, which is like $x^2 - y^2$ (Type 3 (c)), and we get—

$$\begin{aligned}&6(x - y - a)(x - y + a). \\ \therefore &6x^2 - 12xy + 6y^2 - 6a^2 = 6(x - y - a)(x - y + a).\end{aligned}$$

In this case three types of factors have been used.

Exercises 68

- | | |
|----------------------------------|-----------------------------|
| 1. $ab^2c^2 - 5abc + 6a.$ | 7. $(a - m)^2 - (a + m)^2.$ |
| 2. $ab^2 + 9ab - 90a.$ | 8. $abx^2 - 9ab.$ |
| 3. $\pi R^2 - \pi r^2.$ | 9. $4r^2m - 16ms^2.$ |
| 4. $x^2 - (x - y)^2.$ | 10. $q^2 - (r + s)^2.$ |
| 5. $a^2 + 2ab + b^2 - c^2.$ | 11. $3x^2y - 6xy^2 - 9y^3.$ |
| 6. $9x^2 + 24xy + 16y^2 - 4z^2.$ | 12. $32x^3 + 48x^2 - 80x.$ |

SIMPLE EQUATIONS

Example.— $\frac{x + 2}{3} + \frac{2x - 6}{4} = 5.$

Method.—The L.C.M. of 3, 4, 1 (the denominators) = 12.
Multiply each side of the equation by 12.

Then $12 \times \frac{x + 2}{3} + 12 \times \frac{2x - 6}{4} = 12 \times \frac{5}{1},$
 or $4(x + 2) + 3(2x - 6) = 60,$
 i.e. $4x + 8 + 6x - 18 = 60,$
 $10x = 70,$
 $x = 7.$

Reference to this example, with the preceding work on equations, brackets, and multiplication, will remove the difficulties of the following:—

Exercises 69

Solve the following equations:—

- $5x - 6 + 7x = 9x + 13 - 3x - 1.$
- $\frac{6.386}{14.2} = \frac{x}{22.35}.$
- $.365x - 36.39 = 4.78 - 1.43x + 1.25 - 4.265x.$
- $3(x + 5) - 4(x - 7) = 12(x - 4).$
- $x(x + 7) + x(8 - x) - 64 = 3x - 16.$
- $(x + 6)(x - 5) - 10 = (x - 3)^2.$
- $x - \frac{1}{2}x = 4.$
- $3x + \frac{3x}{2} - \frac{5x}{4} = 13.$
- $\frac{x + 1}{4} - \frac{2(x - 1)}{3} = 3.$
- $9.6x - 13.2x - 71.2 = 1.6(1.44x - 0.4).$
- $(x - 4)(x - 3) - 2(x - 5)(x - 4) = (2 - x)(x - 5).$

$$12. 4x^2 = (x - 1)^2 + (x + 2)^2 + 2(x + 4)^2.$$

$$13. \frac{x + 0.5}{2} - \frac{3x - 6.25}{5} - \frac{6}{5} = 0.$$

$$14. \frac{x - 5}{2} - \frac{x - 3}{2} + x = \frac{x - 4}{3} - 2.$$

PROBLEMS LEADING TO SIMPLE EQUATIONS

Example 1.—Two numbers added equal 16.8. Five times the smaller subtracted from the larger gives an answer which is 0.4 times the larger. Find the numbers.

The following *queries* will help to discover a method of solution:—

1. If x is one number, what is the other? Ans. $(16.8 - x)$.

2. If x is the larger number, what is "Five times the smaller subtracted from the larger"? Ans. $x - 5(16.8 - x)$.

3. If this result is ".4 times the larger" number, what is the equation?

$$x - 5(16.8 - x) = .4x.$$

$$\therefore x - 84 + 5x = .4x,$$

$$5.6x = 84,$$

$$x = \underline{\underline{15}}.$$

4. If one number x is 15 by the equation, what is the other? Ans. $(16.8 - 15) = 1.8$.

Example 2.—A man sells a certain number of tables for £12. If he had sold 8 more tables with these for the same money, his price per table would have been only two-thirds of what it formerly was. How many tables did he sell?

QUERIES AND ANSWERS

1. If x is the number of tables, what is the price of each?

Ans. $\pounds \frac{12}{x}$.

2. What is $\frac{2}{3}$ of this price? Ans. $\pounds \frac{2}{3} \times \frac{12}{x} = \pounds \frac{8}{x}$.

3. If he sold x tables and 8 tables for £12, what is the price per table? Ans. $\pounds \frac{12}{x + 8}$.

4. As the price of the table in Query 3 is $\frac{2}{3}$ the price in Query 1, state the equation?

$$\begin{aligned}\frac{12}{x+8} &= \frac{2}{3} \times \frac{12}{x} \text{ (cross-multiply),} \\ 12x &= 8(x+8), \\ 12x &= 8x+64, \\ 4x &= 64, \\ x &= 16. \qquad 16 \text{ tables.}\end{aligned}$$

Exercises 70

Reason the following problems similarly:—

1. The sum of two numbers is 128, their difference is 16. Find them.
2. One number is five times another. If the smaller be subtracted from 20, the difference is equal to four times the smaller. Find them.
3. What number is that which, if added to 60, the result is 16 times itself?
4. x is a certain number. The difference between its third and fourth parts is 2.25. Find x .
5. Two consecutive numbers are such that one-sixth of the smaller and one-fifth of the larger are consecutive numbers. Find them.
6. The length of a rectangular field is $3\frac{1}{2}$ times its breadth. Its perimeter is 5200 yd. Find its length and breadth.
7. The cost of A tons of coal is £5, 10s., and the cost of B tons is £12, 7s. 6d. What relationship exists between the amounts of coal purchased? If one ton is bought for £2.75, find A and B.
8. A is the area of a rectangular surface out of which are punched 8 circular holes of 3 in. diameter. These reduce its area to $\frac{A}{2} + 62.8$ sq. in. Find A.
9. A has £300 and B has £250. How much must A give B so that his share shall be .75 B's share?
10. A certain number of apples are bought for 1s. If 16 more could with these be purchased for the same money, the cost of each apple would be only one-third its former price. How many apples were bought?
11. A certain number of men hire a boat for 10s. If there had been two more men they would each have had to pay 1s. 3d. to make up the hire between them. How many men sailed that boat?
12. 10 coats are sold at such a price as to yield a profit of 5s. on each coat. If the cost of each coat had been one-tenth less, the profit would have been 6s. per coat. Find the selling price per coat.

FACTORS APPLIED TO PRACTICAL WORK

1. The Right-angled Triangle.

Example 1.—The vertical height of a cone is 7 in. and its slant height is 9 in. Find the area of its base.

Method.—ABC is a right-angled triangle.

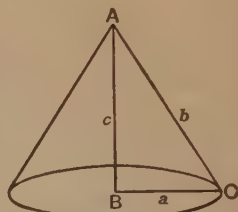


Fig. 45

$$\therefore b^2 = c^2 + a^2, \text{ i.e. } a^2 = b^2 - c^2;$$

$$\text{but } b^2 - c^2 = (b + c)(b - c).$$

$$\therefore a^2 = (b + c)(b - c) = (9 + 7)(9 - 7) = 16 \times 2 = 32.$$

$$\therefore a^2 = 32, \text{ and the radius } a = \sqrt{32}.$$

$$\text{Area of base} = \pi r^2 = \pi \times 32 = 3.1416 \times 32 = 100.5 \text{ sq. in.}$$

2. An Annulus or Ring is the area enclosed by the circumferences of two concentric circles.

In fig. 46, if the radii OA and OB of the concentric circles are respectively R and r,

then area of larger circle = πR^2 ;

also „ smaller „ = πr^2 .

$$\therefore \text{area of annulus} = \pi R^2 - \pi r^2 = \pi(R^2 - r^2) = \pi(R + r)(R - r).$$

If R = 10 in. and r = 6 in., then area of annulus = $\pi(10+6)(10-6) = 64\pi = 201.06 \text{ sq. in.}$

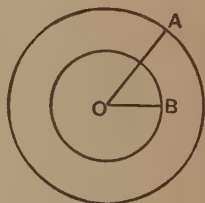


Fig. 46

3. Volume of Hollow Cylinders.

Example 2.—A hollow cylinder has its internal and external diameters 5.4 in. and 6 in. respectively, and its length is 10 in. Find its volume.

Method.—The radii R and r of the annulus at each end are 3 in. and 2.7 in. respectively.

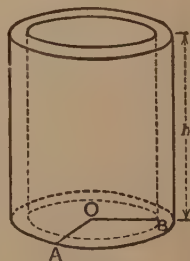


Fig. 47

If cylinder were solid its volume = $\pi R^2 h = \pi \times 3^2 \times 10$.

The cylindrical space has volume = $\pi r^2 h = \pi \times (2.7)^2 \times 10$.

$$\begin{aligned}
 \therefore \text{Volume of metal} &= \pi R^2 h - \pi r^2 h = \pi h(R^2 - r^2) \\
 &= \pi h(R + r)(R - r) \\
 &= \pi \times 10(3 + 2.7)(3 - 2.7) \\
 &= 10\pi \times 5.7 \times .3 = 17.1\pi \\
 &= 53.72 \text{ c. in.}
 \end{aligned}$$

Exercises 71

1. The radii of two concentric circles are respectively 8.2 and 6.8 cm. Find the area of the enclosed annulus.
2. The diameters of two concentric circles are 8.6 and 11.4 cm. respectively. Find the area of the enclosed annulus.
3. The internal and external diameters of an annulus are 6.54 and 9.46 in. respectively. Determine its area.
4. A square garden of 20 yd. side has a square plot of 18 yd. side in its centre. What is the area of the paths around it?
5. A square handkerchief has one side measuring 1 ft. 6 in. It has a hem 1 in. in width running round it. Find the area of the hem.
6. A semicircular platform, diameter 20 ft., has a table 1 ft. wide placed round its semicircumference. Find the area of the table, and the space left available for seating accommodation.
7. A hollow cylinder 4.6 cm. long has its internal and external diameters of 3.82 and 4.18 cm. Find its volume, and the weight of the metal of which it is composed, if a c.c. of the material weighs 8.9 g.
8. A hollow cylinder has a volume of 79.17 c. in., and its external and internal diameters are respectively 7.6 and 6.4 in. Find its length.
9. A ladder 20 yd. long reaches a point on a wall 18 yd. from the ground. Find the distance of the foot of the ladder from the wall.
10. A tangent is drawn to a circle of 3.6 in. radius from a point 4.4 in. away from its centre. What is the length of the tangent?
11. The slant height of a cone is 6.8 in., and the area of its base is approximately 15.21 sq. in. Find its vertical height.
12. The diameter AB of a circle is 15.3 cm., and a point C on its circumference is 4.7 cm. from A. How far is it from B?

DIVISION OF ALGEBRA

Example 1.—It is required to divide $9x^2 + 27x^3y^2 - 4xy^3$ by $3x$.

Method.—

$$\begin{array}{r}
 3x \overline{) 9x^2 + 27x^3y^2 - 4xy^3} \\
 \underline{3x + 9x^2y^2 - \frac{4}{3}y^2}
 \end{array}$$

Analysis of method.—

$$(a) \frac{9x^2}{3x} = 3x; (b) \frac{+27xy^2}{+3x} = +9x^2y^2; (c) \frac{-4xy^3}{+3x} = -\frac{4}{3}y^3.$$

Example 2.—It is required to divide $3x^2 - 17x - 28$ by $3x + 4$.

Method.—

Step 1. $3x + 4 \overline{) 3x^2 - 17x - 28}$ (x Compare 65) 754 (1

To find the first figure in the quotient, determine $3x^2 \div 3x$ just as you determine $75 \div 65$.

$$\begin{array}{r} \text{Step 2. } 3x + 4 \overline{) 3x^2 - 17x - 28} \quad (x \quad 65) 754 (1 \\ \underline{3x^2 + 4x} \qquad \qquad \qquad 65 \\ -21x \qquad \qquad \qquad \underline{10} \end{array}$$

Multiply divisor by number in quotient and subtract.

$$\begin{array}{r} \text{Step 3. } 3x + 4 \overline{) 3x^2 - 17x - 28} \quad (x - 7 \quad 65) 754 (11 \\ \underline{3x^2 + 4x} \qquad \qquad \qquad 65 \\ -21x - 28 \qquad \qquad \qquad \underline{104} \\ -21x - 28 \qquad \qquad \qquad \underline{65} \\ \qquad \qquad \qquad \underline{39} \end{array}$$

Repeat steps 1 and 2 after bringing down the next term.

Method 2.—If the dividend will factorize, proceed thus—

$$\frac{3x^2 - 17x - 28}{3x + 4} = \frac{(3x + 4)(x - 7)}{(3x + 4)} = \underline{\underline{x - 7.}}$$

As in multiplication, always arrange your terms according to the ascending or descending powers of some letter. Note the following, where difficulty is caused by missing terms:—

$$\begin{array}{r} x + y + z \overline{) x^3 + y^3 + z^3 - 3xyz} \quad (x^2 - xy - xz + y^2 - yz + z^2 \\ \underline{x^3 + x^2y + x^2z} \qquad \qquad \qquad \\ -x^2y - x^2z - 3xyz + y^3 + z^3 \\ \underline{-x^2y \qquad \qquad -xyz \qquad \qquad -xy^2} \\ -x^2z - 2xyz + y^3 + z^3 + xy^2 \\ \underline{-x^2z - xyz \qquad \qquad \qquad -xz^2} \\ -xyz \qquad \qquad \qquad +xy^2 + xz^2 + y^3 + z^3 \\ \qquad \qquad \qquad \underline{+xy^2 \qquad \qquad +y^3 \qquad \qquad +y^2z} \\ -xyz \qquad \qquad \qquad +xz^2 \qquad \qquad +z^3 - y^2z \\ \underline{-xyz \qquad \qquad \qquad -y^2z - yz^2} \\ \qquad \qquad \qquad \underline{+xz^2 + yz^2 + z^3} \\ \qquad \qquad \qquad \underline{+xz^2 + yz^2 + z^3} \end{array}$$

Exercises 72

1. Divide $8x^4y^7z^7$ by $6xy^3z^4$ and $-8p$ by $4p$.
2. Divide $12m^3 - 6m^5 - 9m^8$ by $-3m^2$.
3. Simplify—

$(a) \frac{3a^2 + 8a + 4}{3a + 2}.$	$(b) \frac{25a^2 - 9b^2}{5a + 3b}.$
$(c) \frac{x^2 - x - 42}{x - 7}.$	$(d) \frac{x^3 - 2x^2 - x + 2}{x^2 - 1}.$
4. Simplify—

$(a) \frac{12a^2 + 10a + 2}{9a^2 - 15a - 6} \div \frac{8a^2 - 8a - 6}{6a^2 - 21a + 18}.$	$(b) \frac{3s(s^2 + 5s - 24)}{6s^2 + 48s}.$
--	---
5. Divide $3x^3 - 4x - 4$ by $3x + 2$.
6. $(3x^5 - 12x^4 + 17x^3 - 9x^2 + 1) \div (x - 1)$.
7. Divide $a^3 - 3a^2b + 3ab^2 + b^3$ by $a - b$.
8. Divide $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ by $a + b$.
9. $(-a^6 + 3a^2 + 52) \div (2 - a)$.
10. $(x^3 - x^2y - xy^2 - 3x^2 - 2xy + 3x + y^3 - 3y^2 + 3y - 1) \div (x + y - 1)$.

SUMMARY

In this section the processes of multiplication and division of algebraic expressions are explained, together with the factorization of expressions of certain types.

Law of Signs in multiplying and dividing.

Like signs (two plus or two minus signs) give plus.

Unlike signs (a minus and a plus) give minus.

Power Laws.

When powers of any base are *multiplied*, the indices are *added*,

" " " *divided*, " " *subtracted*,

in order to obtain the index of the power of the result.

Factors.

$$\text{Type 1. } (x \pm 3)(x \pm 2) = x^2 \pm 5x + 6.$$

$$\text{Type 2. } (x + 3)(x - 7) = x^2 - 4x - 21.$$

$$\text{Type 3. } (x \pm 3)^2 = x^2 \pm 6x + 9.$$

$$(x + 3)(x - 3) = x^2 - 9.$$

$$\text{Type 4. } (4x - 3)(3x + 5) = 12x^2 + 11x - 15.$$

$$\text{Type 5. } x(a + b) = xa + xb.$$

Factors applied to Formulæ.

1. Right-angled triangle, $a^2 = b^2 + c^2$.
 $\therefore c^2 = (a - b)(a + b)$.
2. Annulus or ring, $A = \pi(R + r)(R - r)$.
3. Hollow cylinder, $V = \pi h(R + r)(R - r)$.

SECTION VII.—CONCLUDING MATTER

The preceding sections of the book have each elaborated certain fixed principles, and the most evident examples have been treated to give these due prominence. Many other points might have been considered in every case, but only with the danger of covering up the fundamental principles on which they were based. It is therefore the endeavour of Section VII to supply additional matter to supplement the foregoing sections, and so to leave the mathematical subjects—arithmetic, mensuration, geometry, &c.—completed to a certain stage.

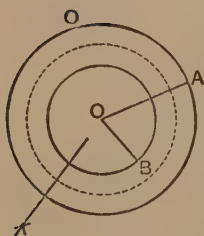
THE SOLID RING

Fig. 48

may be considered to be a prism (if rectangular in section) or a cylinder (if circular in section) which has been curled round and the ends joined.

Let fig. 48 be an indiarubber ring of circular cross section. Cut it across along line marked X and uncurl it. It becomes a cylinder whose length is the dotted circumference of the circle midway in its thickness. If OA and OB are the internal and external radii of the ring, then the diameter of the circular cross section at X is found by taking their difference.

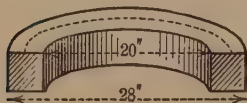


Fig. 49

Example.—Find the volume and surface area of a ring of square cross section whose internal and external diameters are 20 and 28 cm. respectively.

Fig. 49 is half the ring.

Side of square cross section = $(28 - 20) \div 2 = 4$ cm.

Area of cross section = 16 sq. cm.

Diameter of circle midway in thickness of ring = 24 cm.

$\therefore$ Length of prism formed by cutting and unbending ring = 24π .

$\therefore$ Volume of Ring = Area of cross section $\times 24\pi$
 $= 16 \times 24 \times \pi = 1206.4$ c.c.,

and Surface Area of Ring = Perimeter of cross section $\times 24\pi$
 $= 16 \times 24 \times \pi = 1206.4$ sq. cm.

Exercises 73

1. An anchor ring measures 18 in. external diameter. It is 4 in. thick. Find its volume if its cross section is circular.
2. A chain is made of 20 circular links each of circular cross section, and 5 in. and 3 in. external and internal diameters. Find the volume of the chain.
3. Find the surface area of a ring of rectangular cross section. Its internal and external diameters are 1 ft. 4 in. and 2 ft., and it is 2 in. thick.
4. Find the volume of a ring of circular cross section if the internal and external radii are 4.5 and 6.3 cm. respectively.
5. One c. in. of the material of which a ring (circular in section) is composed weighs .03 lb., and its total weight is 17.326 lb. The thickness of the ring is 2 in., find its external diameter.

THE VOLUME OF A PYRAMID AND CONE

I. **Pyramid.**—Fig. 50 represents a triangular prism P divided into three triangular pyramids T_1 , T_2 , and T_3 .

T_1 is made by cutting through the prism from AB to D along the shaded plane.

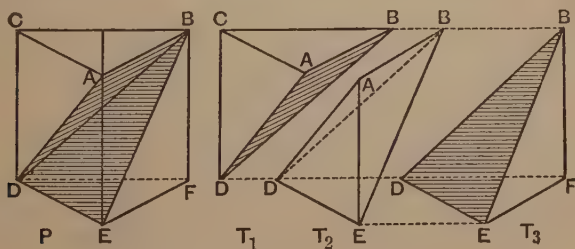


Fig. 50

T_2 and T_3 are cut from the remainder by cutting from B to DE along the shaded plane.

This can be done practically in wood or clay, or three pyramids of cardboard can be made which, put together, will make the prism.

These pyramids can be shown to be equal to one another, for ABD is the common base of T_1 and T_2 , which are of equal altitude, and BDE is the common base of T_2 and T_3 , which are of equal altitude. (*Note.*—Triangular pyramids on equal bases and of equal altitude are equal in volume.)

Thus each pyramid is one-third the volume of the prism.

$$\text{Prism } V = \text{Area of Base} \times \text{Height. } V = Ah.$$

$$\text{Pyramid } V = \frac{1}{3} \text{Area of Base} \times \text{Height. } V = \frac{1}{3}Ah.$$

II. Cone.—*Experiment:* Take a cone of wood or metal and carefully construct a cardboard cylinder into which the cone will just fit, the apex of the cone touching the top circular end. Fill the cardboard box with water or fine sand; pour into a measuring glass and measure. It measures, say, 30 c.c. Next, fit the cone in the box, fill the remaining space with water or sand, and measure again. It measures 20 c.c. Thus the cone's volume is 10 c.c., which is just one-third of the cylindrical space within the box. Thus—

$$\text{Cylinder } V = \text{Area of Base} \times \text{Height. } V = \pi r^2 h.$$

$$\text{Pyramid } V = \frac{1}{3} \text{Area of Base} \times \text{Height. } V = \frac{1}{3}\pi r^2 h.$$

As the result of these and other experiments, the volume of a pyramid or cone will be found to be one-third of the prism or cylinder of equal height and identical equal base. Or—

$$V = \frac{1}{3}\pi r^2 h; V = \frac{1}{3}Ah. \quad (A = \text{area of base.})$$

THE VOLUME OF A SPHERE

Experiment: Take an indiarubber ball, and make a cardboard cylindrical box which will just hold it—the base, lid, and sides just touching the surface of the ball. Fill the box with water or fine sand, pour this into a measuring glass, and take the measure (say 6 fluid oz.). Repeat with the ball placed in the box. The

water or sand measures less (2 fluid oz.). That is, the ball occupies two-thirds of the space of the box. Now the box is a cylinder whose base has the same radius as the ball (sphere), and whose height is double the radius of the sphere.

$$\therefore \text{Volume of Cylinder} = Ah = \pi r^2 \times 2r = 2\pi r^3.$$

$$\therefore \text{Volume of Sphere} = \frac{2}{3} \text{Volume of Cylinder} = \frac{2}{3} \times 2\pi r^3 = \frac{4}{3}\pi r^3.$$

Fig. 51 shows the relationship between the volumes of cylinder (box), sphere (ball), and cone of same height as box.

The cone MDC has a volume $\frac{1}{3}$ of the volume of the cylinder ABCD, and the sphere's volume is $\frac{2}{3}$ the volume of ABCD.

Thus the cone MDC is one-half the volume of the sphere of same diameter as its base and height.

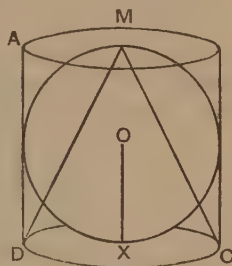


Fig. 51

THE SURFACE OF A SPHERE

Fig. 52 represents a "slice" out of a sphere. Such a slice is termed a frustum. FA is the radius of the upper circle, BE the radius of the lower. C and D are midway between AB and FE.

Since AB approximates to a straight line, ABG is a right-angled triangle, which will be found similar to the right-angled triangle ODC.

I.e. OC is as much longer in proportion than OD as AB is than AG.

Or—

$$\frac{OC}{DC} = \frac{AB}{AG} = \frac{AB}{FE}, \text{ since } AG = FE.$$

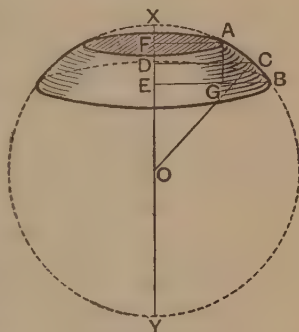


Fig. 52

By cross-multiplication, $OC \times FE = DC \times AB$.

But $2\pi \times DC \times AB =$ the area of the curved surface of the frustum.

Note.— $2\pi DC$ = circumference of circle midway between the ends of the frustum, and AB is the breadth of the frustum.

$$\begin{aligned}\text{Thus if } OC \times FE &= DC \times AB, \\ \text{then } 2\pi \times OC \times FE &= 2\pi \times DC \times AB \\ &= \text{curved surface of the frustum.}\end{aligned}$$

The curved surface of the frustum then equals 2π times OC (radius of the sphere) multiplied by the thickness FE of the frustum.

Suppose 50 such frustums make the sphere, then the curved surface of 50 frustums (i.e. the surface of the sphere) equals 50 times ($2\pi \times OC \times FE$),

$$\text{i.e. Surface of Sphere } S = 50 \times FE \times 2\pi \times r.$$

$$\text{But } 50 \times FE = \text{diameter of the sphere} = 2r.$$

$$\therefore S = 2r \times 2\pi \times r = 4\pi r^2.$$

Thus the surface of a sphere is equal to 4π times the square on the radius.

$$\text{Note.}—\text{Circumference of Circle} = 2\pi r.$$

$$\text{Surface of Sphere} = 2\pi r \times 2r = 4\pi r^2.$$

Exercises 74

1. Find the volume of a cone the diameter of whose base is 8.6 c.m. and whose perpendicular height is 14.7 cm.
2. The slant height of a cone is 28 in., the diameter of its base is 33.6 in. Find its vertical height and its volume.
3. A pyramid has one side of its square base 20.7 in. long and its vertical height 15 in. Find its volume.
4. The volume of a cone is 47.124 c. in. Its base has a diameter of 6 in. Find its vertical height.
5. Find the surface area and volume of a sphere of 3.5 cm. radius.
6. How could you demonstrate to your satisfaction that the volume of a cone is one-third the volume of a cylinder of equal base and height? If the area of the base of a cone is 15.9 sq. in. and its volume is 47.7 c. in., what is its height?
7. What weight of metal is required to cast a spherical knob of 4.5 in. diameter, if 1 c. in. of the metal weighs .26 lb.?
8. Find the area of the ends and curved surface of the frustum of a sphere, given that the diameters of the circular ends are 8 and 12 in., and the distance between them is 4 in.
9. A truncated cone has its circular ends 6 in. and 10 in. in diameter respectively, and the distance between them 4 in. Find the height

and volume of the original cone of which the truncated part is given, the volume of the part removed, and by subtraction the volume of the portion remaining.

10. What is the volume of a triangular prism the triangular ends of which measure 14.7 sq. cm., and whose height is 6.9 cm.? What is the volume of a triangular pyramid of equal base and half the height?
11. A cylindrical box has its circular base 10 in. in diameter and its height 10 in. What is the volume of the spherical ball which will just fit the box? What space in the box is still unoccupied?
12. A cylinder of metal whose height equals the diameter of its base is made into a sphere of diameter equal to the cylinder. With the remaining metal another sphere is made. Compare the volumes of the two spheres.
13. Two spheres have their radii 6 in. and 4 in. respectively. What is the volume of each? Compare them with one another and express the ratio of their sizes.
14. A milk can is a truncated cone. The diameter of its base is 10 in., and the vertical height of the cone of which it is a part is 50 in. Find the volume of milk at present in the can, given that it is 6 in. deep.

SOME FURTHER TRUTHS ABOUT TRIANGLES

Exercises 75.—Experiments

1. Construct a triangle ABC, making AB 3 in. long, the angle A 40° , and the angle B 60° . What is the size of angle C? From C drop a perpendicular to AB. Carefully measure with your ruler the lengths of CA, CD, CB. Multiply the length of AC by the sine of angle A, and CB by the sine of angle B. Compare your answers with the length of CD and state your observations.
2. ABC is a triangle whose sides a , b , c measure 8 cm., 7 cm., and 4 cm. respectively. Carefully measure to the nearest degree with your protractor the size of the angles A, B, C. Does the statement $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ hold true? Substitute and see.
3. A triangle PQR has its base PR divided at X so that PX is 3 cm., XR is 5 cm., and QX perpendicular to PR is 6 cm. Carefully measure the angles P and R, and state their sines. You have two methods of finding the lengths of QP and QR. What are they? Find them by both methods.
4. In sum 3 state x in the following by substituting the values observed.
 - (a) $PQ \sin P = x$.
 - (b) $QR \sin R = x$.
 - (c) What is x equal to in the figure?
 - (d) Do $PQ \sin P = QR \sin R$, and $\frac{PQ}{QR} = \frac{\sin R}{\sin P}$? If so, why?

5. ABC is a triangle right-angled at A. The sides about the right angle are 7.5 cm. and 10 cm. Find the length of the hypotenuse. Obtain from your results: $\sin C$, $\cos B$, $\cos C$, $\sin B$. What relationships do you observe holding among these quantities?
6. ABC is an equilateral triangle of 4 in. side, and AD is drawn perpendicular to BC from A. What are the lengths of AD and DC? Express the sines, cosines, and tangents of the two acute angles of the triangle ACD. What relationships exist among them?
7. The sides a , b , c of a right-angled triangle ABC, right-angled at A, are such that $a^2 = b^2 + c^2$. If both sides of the equation are divided by a^2 , then $\frac{a^2}{a^2} = \frac{b^2}{a^2} + \frac{c^2}{a^2}$, or $1 = \left(\frac{b}{a}\right)^2 + \left(\frac{c}{a}\right)^2$. But $\frac{b}{a} = \cos C$, and $\frac{c}{a} = \sin C$. Thus the equation may be written $1 = \sin^2 C + \cos^2 C$. (Note $\sin C \times \sin C = \sin C$ squared $= \sin^2 C$.) Does this hold true when the angle C is 60° ? Substitute and see.

The following truths are arrived at from a study of Exercise 75:—

1. The sides of a triangle bear the same ratio to one another as the sines of the angles they subtend. Thus, if the triangle is ABC, its sides and angles are related according to the statement $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$. If a perpendicular be dropped from A upon BC, then it is equal to $b \sin C$ and $c \sin B$.

Then $b \sin C = c \sin B$. (This is like the equation $ab = cd$.)

$$\therefore \frac{b}{c} = \frac{\sin B}{\sin C} \quad \left(\text{just as } \frac{a}{c} = \frac{d}{b}\right),$$

$$\text{or } \frac{b}{\sin B} = \frac{c}{\sin C}. \quad \text{Similarly, } \frac{c}{\sin C} = \frac{a}{\sin A}.$$

2. When angles are complementary, i.e. equal 90° if added,

(a) The sine of one angle equals the cosine of its complement; and

(b) The cosine of the angle equals the sine of its complement.

3. $\sin^2 \theta + \cos^2 \theta = 1$. This is followed by working the last question in Exercises 70, and can be verified by using your tables and substituting any angle for θ .

Example.— $\sin^2 30^\circ + \cos^2 30^\circ = 1$.

From your tables—

$$\sin 30^\circ = .5. \quad \therefore \sin^2 \theta = .5 \times .5 = .25.$$

$$\text{Also } \cos 30^\circ = .866. \quad \therefore \cos^2 \theta = .866 \times .866 = .7499.$$

$$\therefore \sin^2 30^\circ + \cos^2 30^\circ = .9999 \\ = 1 \text{ approx.}$$

Exercises 76

1. The two angles A and B of a triangle ABC are 40° and 45° . What is the ratio of the two sides a and b ? If b is 10 in., what is a ?
2. In a triangle ABC it is found that $\sin A$ and $\cos B$ each equal .866. What is the size of angle C?
3. It is given that $\sin^2 \theta + \cos^2 \theta = 1$. Verify the correctness of the statement when $\theta = 70^\circ$ and when θ is 40° .
4. A triangle has its three sides respectively 4, 8, and 6.9 in. If one angle is 30° , find the other angles.
5. Given that $\sin^2 \theta + \cos^2 \theta = 1$, a student conveyed $\cos^2 \theta$ to the opposite side of the equation and obtained the statement—

$$\sin \theta = \sqrt{(1 - \cos \theta)(1 + \cos \theta)}.$$

How was this obtained? Find $\sin \theta$, given that $\cos \theta = .8$.

6. $\frac{a}{b} = \frac{\sin A}{\sin B}$. Find b when $a = 3.2$ in., A is 15° , and B is 45° .
7. In a triangle PQR find p when $q = 4.8$ in., P is 28° , and Q is 57° .
8. The side a of a triangle ABC is 4.8 in., A is 30° , B is 70° . Find C, b , and c .

PROPORTIONAL PARTS

I. Carefully work the following exercises.

Exercises 77

1. Construct an angle BAC of 30° . On AB and AC mark off lengths 2 in. and 1.5 in., and join the points P and Q so obtained. Produce AP to D, making AD 3 in. long. From D draw DE parallel to PQ to meet AC produced in E. From the figure, by measurement, state—
 - (a) The lengths of PQ, DE, and AE.
 - (b) The ratios of AP to AD, AQ to AE, and PQ to DE.
 - (c) In what ratio does P divide the line AD?
 - (d) What is the ratio of AQ to QE?
2. Draw a straight line OX. On it mark off the following lengths,

measuring each from O: OA 3.5 in., OB 4 in., and OC 2.5 in.
State—

- (a) The ratios of OA to AB and AB to CA.
 - (b) In what ratios the points A and C separately divide the line OB.
3. Two similar triangles ABC and AEF are superimposed, the angle A being common to the two triangles. AB, AC, and AE are respectively 4 in., 3.5 in., and 1.5 in. long.
- (a) State the ratios of AE to EB and AF to FC.
 - (b) If AB represents £1, divide it into two parts in the same ratio which AE bears to EB.

II. Fig. 53 should now be easily understood.

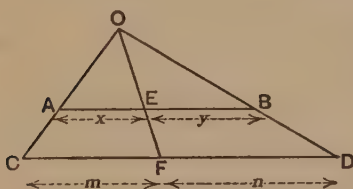


Fig. 53

OAB and OCD are two similar triangles, and any line OEF is drawn dividing them into two sets of similar triangles—OAE, OCF, and OEB, OFD.

Note x , y , m , n are used to represent the lengths of the lines AE, EB, CF, and FD, so that AB is $(x + y)$ units long,

and CD is $(m + n)$ units long.

From prior work done on similar triangles it will be seen that OA bears the same relationship to OC as x does to m and that $(x + y)$ does to $(m + n)$,

$$\text{or } \frac{OA}{OC} = \frac{x}{m} = \frac{x + y}{m + n}.$$

$$\text{Further, } \frac{OB}{OD} = \frac{y}{n} = \frac{x + y}{m + n}.$$

Thus (a) x bears the same relationship to y that m does to n ,
and (b) x and y separately bear the same relationship to m
and n respectively that the whole line $(x + y)$
does to the whole line $(m + n)$.

If $(m + n)$ or CD is 10 in., and x is 2 in., while y is 3 in., then both m and n can be found.

$$\text{For } \frac{x}{m} = \frac{x + y}{m + n}, \text{ i.e. } \frac{2}{m} = \frac{2 + 3}{10}.$$

$$\therefore m = 4, \text{ and consequently } n = 10 - 4 = 6.$$

Example 1.—Divide 10 in the ratio of $\frac{1}{4}$ and $\frac{3}{8}$. (See fig. 53.)

Method.—Draw two lines AB and CD parallel to one another. Make CD equal to 10 in. On AB mark off AE $\frac{1}{4}$ in. and EB $\frac{3}{8}$ in. Join CA and DB, and produce till they meet in O. Join OE, and produce to cut CD in F. Then F divides the 10 in. line in the ratio of $\frac{1}{4}$ and $\frac{3}{8}$. CF and FD will respectively measure 4 in. and 6 in.

$$\text{For } \frac{x}{m} = \frac{x+y}{m+n}, \text{ i.e. } \frac{\frac{1}{4}}{m} = \frac{\frac{1}{4} + \frac{3}{8}}{\frac{10}{m}}, \text{ i.e. } 2\frac{1}{2} = \frac{5}{8}m.$$

$$\therefore m = 4 \text{ and } n = 6.$$

Example 2.—Divide £6 among three people in the proportion of 5, 4, and 3.

Method 1.—Graphical representation. Fig. 54 is constructed as in the last example. A has 5 shares out of $(5 + 4 + 3)$ shares,

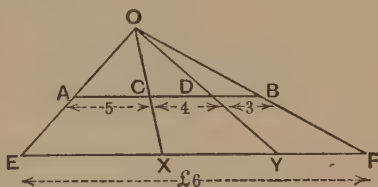


Fig. 54

i.e. A has $\frac{5}{12}$ of £6.

$$\text{For } \frac{AC}{EX} = \frac{AB}{EF}, \text{ i.e. } \frac{5}{x} = \frac{12}{£6}, \text{ or } x = \frac{5}{12} \text{ of } £6 = £2, 10s.$$

B has $\frac{4}{12}$ of £6, or £2, i.e. XY in fig. 54.

C has $\frac{3}{12}$ of £6, or £1, 10s., i.e. YF in fig. 54.

Method 2.—The sum may also be reasoned thus—

If A got £5, B would get £4, and C £3, and £12 would have been distributed. But £6 is to be distributed, of which A must get $\frac{5}{12}$, or £2, 10s.; B must get $\frac{4}{12}$, or £2; and C $\frac{3}{12}$, or £1, 10s.

Exercises 78

1. Divide a line 5 in. long in the proportion of 1 to 1.5 (Graphic method).
2. A line AB 7 cm. long is divided into three parts by the points C, D, such that AC is 2.5 cm. and CD is 3.6 cm. It is required to similarly divide a line 1 dm. long. Show how this can be graphically done, and state the lengths of the parts by carefully measuring them to the nearest millimetre.
3. £10 has to be paid to two men who have worked respectively 17 and 34 days. How should the sum be divided?

4. The perimeter of a triangle is 360 in., and its sides are in the proportion 3, 4, 5. Find the length of each side.
5. Divide £40 among A, B, and C, so that A may get five times what B gets, and C may have as much as A and B together.
6. A bankrupt's assets are £202, 15s. He owes A £157, 10s., B £212, 12s. 6d., and C £440, 17s. 6d. Find what A, B, and C receive, and what the bankrupt pays in the pound.
7. Divide 186s. into an equal number of half-guineas, half-crowns, florins, and sixpences.
8. Divide a line 8 in. long so that the two parts may be in the ratio of $\frac{1}{2}$ to $\frac{1}{5}$.
9. A light rod, whose weight may be neglected, has two weights, $6\frac{1}{2}$ lb. and 7 lb., attached to its two ends. The whole balances about a point of the rod. What is the ratio of the distances of the weights from this point? and how far from the centre of the rod is the point, given that the rod is 68 cm. long?
10. Divide £300 among A, B, and C, so that A may have £50 more than B, and B £25 more than C.

SURDS

A "surd" or "irrational" quantity is one of which the exact root value cannot be obtained. $\sqrt{3}$, $\sqrt{2}$, $\sqrt[3]{a}$ are *surd* quantities. $\sqrt{3}$ is irrational, for $\sqrt{3} = 1.732+$; $\sqrt{4}$ is rational, for $\sqrt{4} = 2$.

A surd may be treated like any other algebraic term.

Thus: *Addition and Subtraction.*

Example 1.— $+5\sqrt{3} + 4\sqrt{3} - 6\sqrt{3} - 2\sqrt{3} + 4\sqrt{3} = 5\sqrt{3}$.
(Compare $5x + 4x - 6x - 2x + 4x = 5x$.)

Multiplication and Division.

$5\sqrt{3}$ means $5 \times$ the square root of 3.

Example 2.—Therefore $5\sqrt{3} \times 4 = 20\sqrt{3}$. (Compare $5x \times 4 = 20x$.)

Example 3.—Also $7\sqrt{x} \div 4\sqrt{x} = \frac{7\sqrt{x}}{4\sqrt{x}} = \frac{7}{4} = 1.75$.

Simplification.

Example 4.— $\sqrt{24} = \sqrt{4 \times 6} = 2\sqrt{6}$, since $4 = 2^2$.

Example 5.— $\sqrt[3]{81} = \sqrt[3]{3 \times 3 \times 3 \times 3} = \sqrt[3]{3^3 \times 3} = 3\sqrt{3}$.

Note $\sqrt{3} = 1.732+$; $\sqrt{2} = 1.414$; $\sqrt{7} = 2.645$. These three values are often used, and should be committed to memory for ready use.

Example 6.—Simplify $\frac{6}{\sqrt{2}}$.

Best method.—

$$\frac{6}{\sqrt{2}} = \frac{6 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2} = 3 \times 1.414 = \underline{\underline{4.242.}}$$

Example 7.—Find the value of $\frac{8}{\sqrt{3}}$ by two methods. Which is preferable, and why?

Method 1.— $\frac{8}{\sqrt{3}} = \frac{8}{1.732} = \underline{\underline{4.618.}}$

Method 2.—

$$\frac{8}{\sqrt{3}} = \frac{8 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{8 \times \sqrt{3}}{3} = \frac{8 \times 1.732}{3} = \underline{\underline{4.618.}}$$

Method 2 is preferable, since the divisor is definite and not approximate in value. Further, this method frequently simplifies the working.

Example 8.—Simplify $\frac{3 + \sqrt{3}}{3 - \sqrt{3}}$.

Method.—

$$\begin{aligned} \frac{3 + \sqrt{3}}{3 - \sqrt{3}} &= \frac{(3 + \sqrt{3})(3 + \sqrt{3})}{(3 - \sqrt{3})(3 + \sqrt{3})} = \frac{(3 + \sqrt{3})^2}{3^2 - (\sqrt{3})^2} \\ &= \frac{9 + 6\sqrt{3} + 3}{9 - 3} = \frac{12 + 6\sqrt{3}}{6} = 2 + \sqrt{3} = \underline{\underline{3.732.}} \end{aligned}$$

Note 1. Aim to rationalize your denominator.

2. $3 - \sqrt{3}$ is like $a - b$ where $a = 3$ and $b = \sqrt{3}$, and $a - b$ becomes $a^2 - b^2$ if multiplied by $a + b$, since $(a + b)(a - b) = a^2 - b^2$. Therefore multiply numerator and denominator by $3 + \sqrt{3}$.
3. The ideas obtained under factors enable the remaining steps to be readily followed.

Exercises 79

1. Find the value of each of the following to four significant figures :—
 $\sqrt{3}$, $\sqrt{6}$, $\sqrt{12}$, $\sqrt{24}$, $\sqrt{27}$, $\sqrt{7}$, $\sqrt{29}$, $\sqrt{147}$.
2. Determine the value of—
 - (a) $6\sqrt{3} - 15 + 4\sqrt{3} - 6\sqrt{27} + 4\sqrt{12} + 47 - 2\sqrt{3}$.
 - (b) $7\sqrt{7} - 3\sqrt{7} + 2\sqrt{7} - 5\sqrt{7} + 4\sqrt{7}$.
 - (c) $7\sqrt{13} \times 3\sqrt{13}$.
 - (d) $129\sqrt{47} \div 3\sqrt{188}$.

3. Add $+5\sqrt{3}$, $+7\sqrt{3}$, $+5\sqrt{12}$, and from the result subtract the difference between $+7\sqrt{3}$ and $+2\sqrt{12}$.
4. Multiply—
 - (a) $4\sqrt{7}$ by $6\sqrt{7}$.
 - (b) $3\sqrt{2}$ by $4\sqrt{8}$.
 - (c) $6 + 5\sqrt{5}$ by $3\sqrt{5}$.
 - (d) $5 - 10\sqrt{7}$ by $4\sqrt{7} + 2$.
 - (e) $8 + \sqrt{3}$ by $8 - \sqrt{3}$.
5. Find the value of—
 $(\sqrt{8})^2$, $(3 - \sqrt{3})^2$, $(\sqrt{7} - 2)^2$, $(\sqrt[3]{27})^2$.
6. Simplify—
 (a) $\frac{8}{\sqrt{2}}$; (b) $\frac{15}{\sqrt{3}}$; (c) $\frac{\sqrt{3} + 1}{\sqrt{3}}$; (d) $\frac{\sqrt{7} - 1}{\sqrt{7} + 1}$; (e) $\frac{2(\sqrt{5} + \sqrt{2})}{\sqrt{5} - \sqrt{2}}$.
7. Determine the approximate value of $\frac{5 + \sqrt{27}}{2\sqrt{3} - 8}$.

VARIABLES AND CONSTANTS

In building up formulæ in mensuration, prominence has been given to “variables”, whose relationships determine the statements made. With advancing knowledge in mathematics it will become clearly understood that every formula can be arrived at by first considering on how many varying considerations the result we desire depends, and then by seeking some relationship existing among these variables.

The following cases we have already considered:—

1. The circumference of a circle varies as the diameter;

$$\text{or } C \propto D.$$

2. The area of a circle depends on the area of the square on the radius. As this square varies in size, so the area of the circle varies;

$$\text{or } A \propto R^2.$$

3. The area of a rectangle or triangle depends on the lengths of base and height;

$$\text{or } A \propto b \text{ and } A \propto h. \quad \text{These can be stated thus: } A \propto bh.$$

4. The time taken to perform a certain work varies inversely as the number of men employed to do it. With more men at work, the work is done in less time, and vice versa;

$$\text{or } T \propto \frac{1}{M} \text{ and } M \propto \frac{1}{T}.$$

5. The distance a mass is from the fulcrum of a lever depends upon its weight. Thus a small weight at a certain distance from the fulcrum will balance a greater weight placed nearer the fulcrum;

$$\text{or } W \propto \frac{1}{D}.$$

6. The *weight* of a log of wood depends upon its *length*, *breadth*, *thickness*, and the average *weight* of a unit amount of its wood (*w*). Thus $W \propto l$, $W \propto b$, $W \propto t$, and $W \propto w$;

$$\text{or } W \propto lbtw$$

7. It is desired to keep the area of a triangle constant; then if the base becomes shorter the altitude must become longer, and vice versa, i.e. the altitude varies inversely as the base;

$$\text{or } A \propto \frac{1}{B} \text{ and } B \propto \frac{1}{A}.$$

We will now put this idea in general form. If there are three quantities P, Q, and R, and P varies directly as the square of Q and inversely as R, then $P \propto Q^2$, also $P \propto \frac{1}{R}$, and $P \propto \frac{Q^2}{R}$.

When the "variables" have been so expressed, the sign of variation, " $\propto$ ", can be changed to the sign of equality, providing that a "constant" be introduced. E.g.—

1. The circumference of a circle is always 3.1416 (or π) times its diameter. Thus $C \propto D$ becomes $C = \pi D$, i.e. $\propto$ becomes = if π is inserted. (π is termed a "constant".)

2. The area of a triangle always varies as its base and altitude, or $A \propto bh$. But experiment shows that half the product of base and altitude gives the area. Thus $A \propto bh$ becomes $A = \frac{1}{2}bh$ by inserting the "constant" $\frac{1}{2}$.

3. If $a \propto \frac{b}{c}$, then $a = \text{some constant} \times \frac{b}{c}$. Let us call this constant k . Then $a = k \times \frac{b}{c}$ or $\frac{kb}{c}$. Similarly, if $P \propto \frac{Q^2}{R}$, then $P = \frac{kQ^2}{R}$, where k is some constant.

In practice, then, it is necessary, for the *stating* of a formula,

to first find all the "variables" on which it depends, and then to experiment to find some relationship among them. In finding this relationship it is often found necessary to put into the formula some number (such as π in $C = \pi D$, and $\frac{1}{2}$ in $A = \frac{1}{2}bh$) which must recur every time any example is worked. This number is "constant", and is so called. It often happens that this "constant" quantity is arranged to be unity.

To Work a Sum in Variation.—Each sum usually contains three statements—

1. A statement as to the variables.
2. A clause containing the result of an experiment with the variables, which will enable you to find the constant.
3. Some unknown to be found from given information.

Example.— x varies directly as a and inversely as y^2 . If x is 3, a is 1, and y is 2, determine y when x is 4 and a is 6.

Method.—

1. $x \propto a$ and $x \propto \frac{1}{y^2}$; then $x \propto \frac{a}{y^2}$. Also $x = k \frac{a}{y^2}$, where k is some constant.

2. Determine the constant thus:

$$x = k \frac{a}{y^2}.$$

Substituting values for x , a , and y in result of experiment given—

$$3 = k \frac{1}{2^2}. \quad \text{Then } k = 12.$$

3. Determine the unknown required thus:

$$x = k \frac{a}{y^2},$$

$$\text{i.e. } 4 = 12 \times \frac{6}{y^2},$$

$$4y^2 = 72,$$

$$y^2 = 18.$$

$$y = \sqrt{18} = \sqrt{9 \times 2} = 3\sqrt{2} = 3 \times 1.414 = \underline{\underline{4.242}}.$$

Exercises 80

1. The circumference of a circle varies directly as the length of its diameter. When the diameter is 25 the circumference is 78.54. Determine the formula and find the diameter when the circumference is 135.8.
2. The time required to do a piece of work varies inversely as the number of men employed. If 20 men can do the work in 6 days, determine by formula how long it will take 35 men to do the work.
3. The area of a parallelogram varies directly as its length and breadth. When the area is 2.88, its breadth is 1.6 and its length 1.8. Set down the formula and show that the constant is unity.
4. The areas of triangles constructed on bases of fixed length vary as their altitudes. The area of a triangle is 15.6 sq. cm. when the altitude is 2.6 cm. Find the area when the altitude is 7.8 cm.
5. Volumes of spheres vary as the cube of their diameters, or $V \propto D^3$.
Given that the constant in this case is $\frac{\pi}{6}$, determine the formula for finding the volume of a sphere. What is the volume if its diameter is 6 in.?
6. a varies directly as the square of b and inversely as c . If a is 4, b is 3, and c is 12, what is c when a is 3 and b is 4?
7. The total weight of a substance varies directly as its relative density. A cylinder of copper, whose relative density is 8.9, weighs 15.6 lb. What would be the weight of an iron cylinder of equal volume, given that the relative density of iron is 6.2?
8. The current of electricity flowing in a circuit varies directly as the electric force driving it along, and inversely as the resistance overcome. If the current is 4.5 amperes, the resistance overcome is 5 ohms, and the electric force is 22.5 volts. Find the resistance when a force of 180 volts promotes a current of 10 amperes.
9. The distance which a body falls from rest varies directly as the square of the time during which it has fallen. When it has fallen during two seconds the distance travelled is 64 ft. Find how long a stone will take to fall from a height of 1600 ft. Also, if a pebble takes 8 sec. to fall from the top of a cliff to the sea, estimate the height of the cliff.
10. The volume occupied by a gas is inversely proportional to the pressure to which it is subjected. When the volume is 20.3 c.c., the pressure is found to be equal to the weight of 80 g. What will be its volume when the pressure is increased to 100 g.?

Concluding Exercises

A

1. Find the value, to four significant figures, of—

$$36.08 \times 15.39 \text{ and } 36.08 \div 9.315.$$

2. Express 8 yd. 1 ft. $9\frac{1}{2}$ in. as feet and the decimal of a foot.

3. Write down the values of the following:—

$$\sin 56^\circ, \cos 18^\circ, \tan 43^\circ, \text{ and } \sin^{-1} .866, \cos^{-1} .7071, \tan^{-1} 1.$$

(Note $\sin^{-1}x$ means the angle whose sine is x .)

4. Write down algebraically: Five times the sum of the squares of a and b subtracted from ten times the cube of c . Divide the whole by three times the product of a and the difference of b and c .
5. Determine x in the following:—

$$\frac{1.6}{x} = \frac{3.85}{2.46}$$

B

1. The length of a rectangle is 8.93 m. and its breadth is 5.36 m. Express its area in square decametres, and also say how broad a rectangle of equal area must be whose length is 6.75 m.
2. Two sides of a triangle are respectively 3.6 and 4.8 in., and the included angle is 45° . Find the area of the triangle.
3. The area of a circle is 238.9 sq. cm. Find its diameter.
4. What is the weight of a telegraph wire of diameter 2.5 mm. and length 1 Km., if 1 c.cm. of the copper of which it is made weighs 9 g.?
5. Find the volume of lead in a pipe 20 ft. long whose external diameter is 2 in., and the thickness of lead in the pipe is $\frac{1}{2}$ in.

C

1. Two similar triangles have sides, similarly placed, 5 cm. and 4 in. respectively. What is the length of the side of the second triangle similarly placed to a side 6.3 cm. long?
2. Find the length of a belt passing over two wheels, each 1 ft. 6 in. in diameter, if their centres are 8.6 ft. apart.
3. A man invests equal amounts of money in the 3-per-cents at 87 and the $4\frac{1}{2}$ -per-cents at 92. He finds he can purchase 5 complete shares in the first. How many has he in the second? (Neglect brokerage.)
4. A man obtains £3 interest on the loan of £150 for 146 days. At what rate is the interest paid?
5. What is the percentage gain in selling an article for £1, 19s. $4\frac{1}{2}$ d. which cost 15 half-crowns?

D

1. Express £6, 18s. $7\frac{1}{2}d.$ as pounds and the decimal of a pound.
2. 7 articles cost x pence. Had one less been sold for the same price each would have cost 1d. more. Find their price.
3. Factorize: (a) $x^2 - 7x + 12$; (b) $6a^2 - ap - 35p^2$; (c) $27a^2 - 12b^2$.
4. Multiply $x^3 - 3x + 1$ by $2x^3 + x^2 - 5$.
5. The shadow of a tree is 68 ft., and the altitude of its top from the end of the shadow is 58° . How high is the tree?

E

1. Divide 14 ft. 7 in. into two parts, one of which is two and a half times the other.
2. Express 4 yd. 1 ft. $3\frac{1}{2}$ in. as yards and the decimal of a yard.
3. What number raised to the third power is 729? Express also its fifth power.
4. Show, by a graphical demonstration, that $5\frac{1}{2} \times 5\frac{1}{2} = 30\frac{1}{4}$.
5. Express as approximate numbers to 4 significant figures, and work the following:—
 (a) 3.14159625 times 6.83972.
 (b) $14684 \div 4.30095$.

F

1. It is found that the measure of a straight line is 27.6 in. and 70.1 cm. Determine the number of centimetres which make 1 in. and the number of inches to a metre.
2. Measure as accurately as possible the length and breadth of your paper in centimetres and inches. Determine its area in square centimetres and square inches, and hence determine the number of square centimetres which equal 1 sq. in.
3. What is the error per cent in writing 3.158 as 3.15 instead of 3.16?
4. Olive oil is .922 times the weight of water, bulk for bulk, and sulphuric acid is 1.84 times the weight of water. Determine the weight in grams of 2 litres of each liquid.
5. A mould is made for casting 50-g. weights of a metal 9 times the weight of water, bulk for bulk. If the mould is cylindrical in shape and 2 cm. deep, what is its diameter?

G

1. If $4d = \frac{5(d^2 + a^2)}{3d}$, find the ratio of d to a .
2. Determine the square root of 568.327.
3. $H = .24C^2Rt$. Determine t if H is 316, C is 5, and R is 4.
4. Find the length of the diagonal of a rectangle whose sides are 22.8 cm. and 30.4 cm.
5. Determine the area of a triangle the base of which is 14.7 in., and whose altitude is 8.6 in.

H

1. The diameters of two wheels are 3 ft. 6 in. and 1 ft. 8 in., and one is driven by a belt passing over the other. If the smaller is making 1000 revolutions per minute, at what rate is the larger one travelling?
2. How many grams of water will a cylindrical can hold if it has a diameter of 18.4 cm. and a length of 4.3 dm.?
3. A hollow cylinder is 8 cm. in diameter. The thickness of the metal is uniformly 5 mm., and its volume is 117.81 c.c. Find its length.
4. Express algebraically: 8 times the square root of a divided by the square of the sum of a and b . Subtract your result from the sum of $3a$ and $5b$. Find the value of this expression when $a = 64$ and $b = 20$.
5. Work graphically, by means of similar triangles, the sum: If 20 articles cost £8, 10s., what should be the price of 14 at the same rate?

I

1. The interest of A in a business is 15%. The total profits made during one year were £350. Find the amount of A's share.
2. By selling shares at 15s. each a man lost 20% on the money he invested. What profit would he have made if he had not sold until the shares would each realize 22s.?
3. A broker receives £15, 2s. 6d. for selling shares for a client in the $4\frac{1}{2}$ -per-cent stock at 94 $\frac{3}{8}$. How much did the broker pay over to the client? (Brokerage $\frac{1}{8}$ th per cent.)
4. A metre ruler is balanced about its 50 cm. mark. A body of unknown weight placed on the 84.6 cm. mark balances a 50-g. weight placed on the 38.9 cm. mark. Determine the weight of the body.
5. The arms of the lever of a safety valve of .4 sq. cm. area are 5 cm. and 2 dm. respectively, and the weight at the end of the longer arm is 4 Kg. Find the pressure of steam in the boiler requisite to open the valve.

J

1. What is the mean proportional of 6.8 and 13.6?
2. One side of a right-angled triangle is 9.8 in. long, and subtends an angle of 58° . Find the lengths of the other two sides.
3. Find the total surface of a cone the diameter of whose base is 7.6 in., and whose vertical height is 10 in.
4. $\cos(A + B) = \cos A \cos B - \sin A \sin B$. Prove this statement to be correct when $A = 58^\circ$ and $B = 37^\circ$.
5. What is the distance of a point from the centre of a circle of 15.9 cm. radius, if the tangent drawn from it to the circle is 21.2 cm. long?

K

1. Simplify $3ab(a - b) + 3bc(b - c) - 3ac(a - c) - (a + b - c)^3$.
2. What must be taken from $7x^2 - 6xy$ to leave $8y^2$?
3. Solve the equations—
 (a) $\cdot 03x - \cdot 175x = \cdot 0125x - \cdot 0315$.
 (b) $6x(x + 8) - 3(x - 2) = 6(x + 7)^2$.
4. Divide $x^4 - 12x^3 + 54x^2 - 108x + 81$ by $x^2 - 6x + 9$.
5. Find the square root of $23587\cdot 68$.

L

1. What is the volume of a square pyramid each edge of which measures 20 in.?
2. If the square of x varies directly as the product of y and z and inversely as m , and x is 15 when y is 4, z is 18, and m is 0.5, find m when x is 12, y is 6, and z is 20.
3. Two sides of a triangle measure respectively 18 and 15 in., and the angle which the latter subtends is 40° . Find the dimensions of the other angles to the nearest value you can.
4. Divide £300 between A and B in the ratio of $\frac{1}{7}$ to $\frac{1}{5}$.
5. Simplify—
 (a) $\frac{6 + \sqrt{3}}{6 - \sqrt{3}}$; (b) $(8 + \sqrt{3})^2 + (8 - \sqrt{3})^2$; (c) $\frac{9\sqrt{8} - 3\sqrt{18}}{\sqrt{18} + \sqrt{8}}$.

MATHEMATICAL TABLES AND CONSTANTS

1 in. = 25.4 mm.

1 gal. = .1604 c. ft. = 10 lb. of water at 62° F.

1 knot = 6080 ft.

1 lb. avdp. = 7000 gr. = 453.6 g.

1 c. ft. of water weighs 1000 oz. or 62.3 lb.

1 radian = $57\cdot 3$ deg.

RELATIVE DENSITIES

Methylated spirit	...	.831	Glass (crown)	...	...	2.6
Olive oil	...	.922	„ (flint)	...	...	3.3
Wood (ash)	...	.75	Gold	...	...	19.3
„ (oak)	...	.91	Gold coinage	...	...	17.5
„ (pine)	...	.56	Ice	...	...	.92
Wax	...	.95	Iron (cast)	...	...	7.2
Brass	...	8.43	„ (wrought)	...	...	7.7
Bronze coinage	...	8.64	Silver	...	...	10.5
Copper	...	8.95	Tin	...	...	7.3
			Zinc	...	...	7.1

E.g. copper, relative density, 8.95, i.e. 1 c. ft. of copper weighs 8.95 times the weight of 1 c. ft. of water.

ANSWERS

Ex. 1—MENTAL

1. 23. 2. $4\frac{7}{10}$ or 4·7. 3. 46 mm. 4. $6\frac{8}{10}$ or 6·8 cm.
 5. 97 mm. 6. 100, 250, 346. 7. 1000; 4 m. = 4000 mm.,
 3 dm. = 300 mm., and 6 cm. = 60 mm. ∴ 4 m. 3 dm. 6 cm. = 4360 mm.;
 2048 mm.; 306 mm.

Ex. 2—MENTAL

1. 46. 2. 1·5, 62·5, 438·5, 1·5, 62·5. 3. 1·5, 31·5, 60, 460.
 4. 140, 2·6, ·8, 240. Method A—By dividing and multiplying by *ten*.
 The decimal point is moved one to the left in dividing by 10, and one to
 the right in multiplying by 10. 5. 604, 680, 434, 620.
 6. 73·6, 760, 804. 7. 3·65, 45·68, 5·37. 8. 638, 476, 1430, 1687.
 Method B—By dividing or multiplying by 100, i.e. by moving the decimal
 point two to the left or two to the right respectively. 9. 6432·5 m.,
 64325 dm., 643·25 Dm., 6·4325 Km. 10. 4389 mm., 438·9 cm.,
 43·89 dm., 4·389 m. 11. 5040 m., 603·6 m., 4·003 Dm.
 12. 540·2 cm., 350 cm., 4067 cm.

Ex. 3—MENTAL

1. 6·53 g. 2. 1000 g. 3. 7·5 francs or 7 fr. 50. 4. 400 cg.
 5. 225, 5·625 fr. or 5 fr. 625. 6. 12·9 m. 7. 368 cm.
 8. £7, 5s. 8d.—4 farthings; 6890 farthings. 9. 1 fr. 35.
 10. 680, 486, 27. 11. ·72 fr. or 0 fr. 72. 12. 1·14 Dm.

Ex. 4

2. ·75, ·375, ·0625, ·4375, ·35, ·12, ·575. 3. ·375, ·1875, ·09375,
 ·046875, ·075, ·0075. 4. ·0625, ·00625, ·3125, ·15625, 1·25.
 5. ·5833+, ·1875, ·4285+, ·3333+. 6. $\frac{7}{8}$. $\frac{7}{4} = 1·75$ ∴ $\frac{7}{8} = ·4375$.
 ∴ 3 lb. 7 oz. = 3·4375 lb. 7. 3·75 ft.; 6·375 m.; 7·5s.; 6·625 qr.
 8. 23·4166+ ft., 8·8611 yd. 9. 4·875 qt. 10. £1·167, £4·631,
 £18·676, £·991, £·03. 11. ·3671875 lb.

Ex. 5

1. Ruler work. 2. 82 mm. ·82 dm. 3. By moving the decimal
 point one place to the left. 4. 6·5 dm. or ·65 m.; 4·8 cm., 48 mm. or
 (0 200) 173 12

- 48 dm.; 6·9 mm., 650 m., 48 m., 6·9 m. 5. 6·58 m., 658 cm., 65·8 m.
 or 6·58 Dm., 658 m. or 6·58 Hm. 6. 467 mm. 7. 5·75 yd.
 8. 10·7916 + ft. 9. 3·8666 + ft. 10. 21·25 in. 11. 6 fr. 50.
 12. 6·425 m. 3·855 m., i.e. 6·43 m. and 3·86 m. to the nearest cm.
 13. 189 cm. 14. 5·9 m.

Ex. 6

1. 81, 25, 512, 81, 10,000, 1,000,000, 2. 2. 3, 5, 4, 3, 6, 4.
 3. 4, 3, 100, 30. 4. 4, 6, 8, 9. 5. $680, 6·8 \times 10^2$; $36, 3·6 \times 10$;
 $57·4, 5·74 \times 10$; $467·8, 4·678 \times 10^2$; $6800, 6·8 \times 10^3$; $360, 3·6 \times 10^2$;
 $574, 5·74 \times 10^2$; $4678, 4·678 \times 10^3$; $47560, 4·756 \times 10^4$; $1283·7, 1·2837$
 $\times 10^3$; $475600, 4·756 \times 10^5$; $12837, 1·2837 \times 10^4$. 6. $6·83 \times 10^2$;
 $5·436 \times 10^3$; $8·36542 \times 10^6$; $5·007 \times 10^3$; $5·0 \times 10$; $2·08 \times 10^2$; $2·56 \times 10^5$.
 7. ·123 m. 8. 600, 68, 538·6, 123·65, 12365. 9. 25, 26, 14, 60.

Ex. 7

1. 16 sq. in. 2. 9 sq. in. 64 sq. cm. By raising the length of the side
 to the second power. 3. The area of a square is determined by multi-
 plying the length of its side by itself, i.e. by raising the length of the side
 to the second power. 4. 20 sq. in. By multiplying the length and the
 breadth together. 5. The area of a rectangle in square measure equals
 the product of the length and breadth of the rectangle in linear measure.
 6. 1 sq. ft.; 9 sq. ft. = 1 sq. yd. or 3^2 ft. = 1 sq. yd. 7. 1 sq. in.,
 12^2 or 144 sq. in. = 1 sq. ft. Half the side of the square represents
 6 in., but the square on this only represents $\frac{1}{4}$ of a square foot. Thus
 $\cdot 5 \times \cdot 5 = \cdot 25$. 8. 100 sq. mm. = 1 sq. cm., 100 sq. cm. = 1 sq. dm.,
 100 sq. dm. = 1 sq. m., and so on; 154625 sq. cm. 9. $\frac{1}{4}$ sq. in. or
 $\cdot 25$ sq. in. $30\frac{1}{4}$ sq. yd. = 1 sq. p. 10. (i) $\cdot 5$ sq. in. = 1 in. $\times \cdot 5$ in.;
 (ii) $\cdot 25$ sq. in. = $\cdot 5$ in. $\times \cdot 5$ in. or 1 in. $\times \cdot 25$ in.; (iii) $\cdot 0625$ sq. in.
 = $\cdot 25$ in. $\times \cdot 25$ in. 11. (a) $\cdot 2$; (b) $\cdot 04$; (c) $\cdot 2 \times \cdot 2 = \cdot 04$; (d) hundredths;
 (e) $\cdot 5, \cdot 25, \cdot 5 \times \cdot 5 = \cdot 25$, hundredths; (f) AM = 6·5 cm., $\therefore \cdot 65$ of AB;
 (g) By counting 36 sq. cm., 12 half square centimetres, and 1 quarter
 square centimetre, $\therefore$ area = 42·25 sq. cm., i.e. 4225 sq. dm.; (h) $\cdot 65 \times \cdot 65$
 = 4225 sq. dm.

Ex. 8

1. 465·9375 sq. ft. 2. 243·36 sq. cm., 79·21 sq. cm., 164·15 sq. cm.
 3. 34 sq. yd. 4. £2, 16s. 6d. approximately. 5. 49·375 sq. ft.;
 16s. $5\frac{1}{2}$ d. 6. 2592·80 sq. cm., 25·928 sq. m. 7. 18·4989 sq. m.
 8. 191 fr. 625. 9. 1 fr. 60. 10. 5·814288 sq. m.

Ex. 9

1. and 2. Varied. 3. $\frac{1}{32}$ in. = 0·0312 in. and 1 mm. = 0·03937 in., $\therefore$ (b) is
 more likely to be accurate. 4. 3 ft. 2 in.; 4 ft. 8 in.; 7 ft. 11 in.; 3 ft. 5 in.
 5. £4, 13s. 6d. 6. 5 yd. 2 ft. 4 in. 7. 3s. 8. 5 out of 4865,

- i.e. 5 out of 5000 approximately, i.e. .1 per cent. 9. $3\cdot7638 +$ yd. and $3\cdot75$ yd. Error = $\cdot0138$ yd. 10. 2 mm. and 20 mm. 10 times.
11. (a) $\pounds5\cdot935$; (b) $3\cdot833$ yd.; (c) $13\cdot67$ ft.; (d) $5\cdot302$ m.; (e) $530\cdot2$ cm.; (f) $133\cdot5$ s.; (g) $4\cdot75$ oz.; (h) $72\cdot03$ g.; (j) 6501 cg. 12. (a) $3\cdot66$ ft.; (b) $3\cdot33$ ft.; (c) $\pounds7\cdot87$; (d) $4\cdot33$ s.; (e) $7\cdot25$ d.; (f) $8\cdot37$ m.; (g) $\cdot67$ yd.; (h) $8\cdot17$ in.
13. (a) 24 and 28, possibly 27; (b) 200 and 201; (c) 23; (d) Two hundred thousand. 14. (a) Should read $3\cdot66$ approximately, because 8 thousandths is 1 hundredth approximately; (b) should read $101\cdot8032$, $2\cdot54 = 3$ approximately and 40×3 only yields hundreds; (c) should read $53\cdot02$ dm.; 1 mm. = $\frac{1}{100}$ dm. 15. $\pounds3$, 16s. $10\frac{1}{2}$ d.

Ex. 10

1. $8\cdot449$, $11\cdot149$, $13\cdot478$. 2. $107\cdot33$, $65\cdot95$, $116\cdot22$. 3. $1252\cdot1$, $545\cdot9$, 726 . 4. $635\cdot2$, $871\cdot2$, $173\cdot4$. 5. $1\cdot171$, $1\cdot537$, $1\cdot7326$. 6. $\cdot09109$, $\cdot3435$, $\cdot05073$. 7. $\cdot2484$, $\cdot3977$, $2\cdot593$. 8. $\cdot02124$, $8\cdot037$, $2\cdot266$. 9. $26\cdot99$, $6\cdot061$, $32\cdot67$. 10. $863\cdot8$, $199\cdot5$, 1774 . 11. 21066 , $\cdot06968$, $\cdot005864$. 12. $31\cdot74$, $\cdot04237$, $\cdot02132$.

Ex. 11A—MENTAL

1. 12, 144, 12, 1728. 2. 1000. 3. 1000, 1000×1000 , i.e. 1,000,000. 4. $1728 + 864 = 2592$ and 864 sq. in. 5. $4\cdot25$ c.c., $25\cdot25$ c.c. mark. 6. 100 c.dm. or $\cdot1$ c.m. 7. 1000 c.c. = 1 litre. 8. 6000 oz., 1000 oz. 9. 1000 g., 10^6 g. 10. $25\cdot6$ g. water, $19\cdot2$ g. oil. 11. 108 g. 12. $25,200$ oz.; 4 ft.

Ex. 11B

1. $235\cdot8$ lb. or $2\cdot1$ cwt. 2. $22\cdot5$ lb. 3. $\cdot06076$ c.m. 4. 3 c. ft. 221 c. in. 5. $6\cdot76$ g.

Ex. 12

1. Various. 2. $58\cdot1$ sq. cm. 3. $17\cdot34$ sq. in. 4. $19\cdot17$, $4\cdot535$. 5. 278 lengths ($278\cdot\frac{3}{4}$). 6. 27,161. 7. $43\cdot26$. 8. $42\cdot7269$ sq. m., $66222\cdot43$ sq. in. 9. $3093\cdot75$ ft. 10. 2, 3, $13\cdot83$. 11. 500 sec., i.e. 8 min. 20 sec. 12. (a) $3 \times 4 = 12$. $\therefore$ First figure must be *tens* and here it is *hundreds*. (b) $60 \div 10 = 6$. Thus the answer should read *units*, $3\cdot7$ not $\cdot37$. (c) $9\frac{1}{2} = 38$ farthings = $\pounds0\cdot39$ approximately. $\therefore \pounds3$, 6s. $9\frac{1}{2}$ d. = $\pounds3\cdot339$. (d) $\cdot387$ is $\cdot4$ approximately. (e) $\frac{1}{4} = \cdot25$. $\therefore \frac{1}{400} = \cdot0025$ and $\frac{1}{400} = \cdot0025$. $\cdot025$ is therefore ten times greater than the fraction $\frac{1}{400}$.

Ex. 13

1. 21, 96, 48, 168, 54, 72, 24. 2. 18, 144, $1\cdot935$. 3. (a) 25; (b) $1\cdot21$; (c) $\cdot376$; (d) $1\cdot976$. 4. $2\cdot391$. 5. $\frac{2}{3} = \frac{4}{6}$. $\therefore \frac{2}{3} \times \frac{3 \times 6}{1} = \frac{4}{6} \times 3 \times 6$. $\therefore 2 \times 6 = 4 \times 3$. If $\frac{a}{b} = \frac{c}{d}$

then $ad = bc$. The divisors reappear on the other side of the equals sign as multipliers.

6. $a = 3.5$; $d = 8.9$. 7. (a) 1.786;
 (b) 2.428; (c) 48.24; (d) .265; (e) 1.428; (f) .0106; (g) 1.92×10 .
 (h) 1.0585.

Ex. 14

1. 6. 2. (a) 756; (b) 1200; (c) 6; (d) 0; (e) 2160; (f) 6; (g) 0.
 3. 6, 6*a*, 38.4, 17.7, $12ab^2$. 4. 178.53. 5A. 26.15. 5B. 15.9.
 5c. 106.4 or 1.064×10^2 . 6. 57.2.

Ex. 15A

1. 20.625. 2. 10196.4 sq. m. 3. 9.7 ft. 35 boards one way or
 20 boards the other way. 4. 13 sq. dm. 32 sq. cm. 25 sq. mm.
 5. £4, 5*s*. 2½*d*. 6. 37.9 m. 7. 6.825 cm. 8. 129.032 sq. cm.,
 2.54 cm., 50.8 cm., .508 m.

Ex. 15B

1. £13, 8*s*. 3½*d*. 2. £35. 3. 534 bricks. 4. 444 bricks.
 5. 200. 6. 74.2 in. 742 sq. in. 7. Perimeter of room
 $= 2 \times 16 + 2 \times 14 = 60$ ft. Area of walls $60 \times 9 = 540$ sq. ft.
 8. 490 yd. 8 in. 9. £14, 8*s*. 9*d*.

Ex. 16

1. Use fig. 1. 2. The diagonals bisect one another at right angles.
 3. Draw equilateral triangles on both sides of AB and join their vertices.
 5. CD bisects $\angle ACB$. Cut off equal arms from the angular point, join
 these points and on the side remote from the angle strike arcs as if to
 construct an equilateral triangle.

Ex. 17

1. Perpendicular = 2.6 in. (a) Into a rectangle; (b) $4 \times 2.6 = 10.4$
 sq. in.; (c) length $\times$ perpendicular height. 2. (a) A rectangle equal in
 area. (b) Parallelograms of equal base and height are equal in area.
 3. Two equal triangles. Yes; but the triangles would differ in shape.
 4. (a) A parallelogram is formed; (b) it is twice the area of the triangle.
 5. 4 times; $\frac{9}{4}$ or 2.25 sq. in. 6. The answers are approximately equal,
 3.13 to 3.15 according to the accuracy of the measurements. The area is
 found by multiplying the diameter by this number (π), since the circum-
 ference is always this number times the diameter. 7. The larger the
 circle the more accurate, because any error made in measurement will be
 proportionately smaller than in the case of a small circle. 8. In the
 case of the largest circle drawn. Because the area of the circle is always
 a certain number of times (π times) the square on the radius. 9. .78 to
 .8. One-fourth of the results of question 8. The square is four times
 larger.

Ex. 18—MENTAL

1. 32.5 sq. in.
2. 15.75 and 31.5 sq. in.
3. 10.5 in.
4. 42 sq. cm.
5. (a) $6\frac{2}{7}$ in., $9\frac{3}{7}$ cm., 7.85 dm.; (b) $6\frac{2}{7}$, $18\frac{3}{7}$, $12\frac{4}{7}$, $50\frac{3}{7}$, $31\frac{3}{7}$.
6. 2.25 , 1.44 , 9 , 16 sq. in. (squares); 7.07 sq. in., 4.52 sq. in., $28\frac{2}{7}$ sq. in., $50\frac{2}{7}$ sq. in. (circles).

Ex. 19

1. 30.34 sq. cm.
2. 10 in.
3. (a) 7.4 sq. in.; (b) 8.75 sq. in.
- (c) 50 sq. cm.
4. 100.26 sq. cm.
5. 48.9 m.
6. 7.25 .
7. 6.125 sq. in.
8. 2.6 in. (altitude); 7.8 sq. in.
9. 32.5 mm.;
- 10.56 sq. cm.

Ex. 20

1. 7.382 sq. in., 37.134 sq. cm., 51.45 sq. m., 3.267 sq. ft.
2. 680.6 ft. per minute.
3. 3181 m. per minute.
4. 224.1 revolutions per minute.
5. 4.96 ft.
6. 14.52 sq. cm., 24.63 sq. m., 15.1 sq. ft.
7. 40.72 sq. in., 69.39 sq. in., 169.7 sq. in.
8. 9.325 and 4.66 m.
9. Diameter = 186.8 in., 27413 sq. in. (approximately).
10. Square 12.25 sq. in. Each triangle 3.06 sq. in. Circles 9.619 sq. in. (inscribed); 19.24 sq. in. (circumscribed).

Ex. 21

1. (a) 2.09 c. ft.; (b) $.01$ c. ft.
2. (a) 6393.7 sq. in.; (b) 24710.4 sq. in.; (c) 4993 sq. in.
3. $14,365,002$ c.c.
4. (a) 1.673 c.dm. (b) $.014863$ cm.
5. 3.6075 cm., 3607.5 c.dm., 3607.5 litres.
6. 2705.625 Kg.
7. 7800 Kg.
8. 15.75 c. ft.
9. 29160 oz. or 16 cwt. 30.5 lb.

Ex. 22

1. 67 c. ft. 54 c. in.
2. 54.48 c. ft.
3. 439.3 c.c.
4. 20 dm. or 2 m.
5. 1.25 sq. ft., 1.26 ft.
6. 1000 in.
7. 181 c. ft. approx., $90,500$ oz. or 2.5 tons approx.
8. Edge, 17 in.;
- volume, 4913 c. in.; 15967.25 oz. or 8.9 cwt.
9. 7 in. and 20 cm.
10. (a) 204.6 c. in.; (b) 118.3 c. in.; (c) 52.92 c. in.
11. (a) 44.8 in.;
- (b) 8793.1 cm.; (c) 2.3 in.
12. 3657.7 c. in.; 1060.7 lb.

Ex. 23

1. (a) 1.715 lb.; (b) 14.45 lb.
2. 64 c. in.
3. 15 c.c.
4. 50 c. in., $.208$ sq. in.
5. 242 bullets.
6. $.932$ c. in.
7. 14.5 lb.

Ex. 24

1. (a) $\frac{ax^2}{y^3}$; (b) $\sqrt{c^3x^2}$; (c) $\frac{ab}{m} = 3xy$.
2. (a) $\sqrt[3]{abc}$; (b) $a^2 = bc$;
- (c) $x^4y = 15\frac{\sqrt{y}}{x}$.
3. 1.0585 ohm resistance.
4. 11.79 horse-power.

5. $F = \frac{a \times b}{d^2} = \frac{ab}{d^2} = 3$. 6. 1.05 amps. 7. 12.25.
 8. 196 ft. 9. 252.5 ft. per second. 10. 448.9 c.c. 11. 1350.
 12. 59.9 sq. in. or 60 sq. in. approximately.

Ex. 25

1. $\frac{3}{8}$ or .375. 2. $\frac{1}{4}$ or .25, and $\frac{a}{x}$ 3. 2s. 6d., 12s., 13s. 4d.,
 17s. 6d., 30s. 4. $\frac{1}{12}$ of 5s. = 5d. 5. $\frac{3}{4}$ or .75.
 6. $146.3 = \frac{1}{2} \times 6 \times \text{base}$. $\therefore \text{Base} = 48.7$.
 $438.9 = \frac{1}{2} \times 18 \times \text{base}$. $\therefore \text{Base} = 48.7$. } Statement is correct.
 Ratio of areas, $\frac{1}{3}$; ratio of altitudes, $\frac{1}{3}$. 7. (a) $\frac{1}{2}$; (b) Area ABC
 $= \frac{1}{2} \times \frac{1.5}{1} \times 2 = 1.5$ sq. in. Area ABD = 3 sq. in. Ratio of areas, $\frac{1}{2}$.

- (c) $\frac{a}{b}$; (d) $\frac{c}{d}$. (e) That the altitudes bear the same ratio to one another as the areas. 8. (a) 2.3, 5 in., 4.6. (b) $\frac{1}{2}$ in each case. (c) They are equal. (d) The sides of the triangles similarly placed are proportional; otherwise, that the ratio existing between two similarly placed sides of the triangles is equal to the ratio of the other two pairs of similarly placed sides.
 9. $\frac{a}{b}$, $\frac{a}{c}$, $\frac{m}{xy}$, $\frac{x+d}{x+2d}$. 10. $\frac{a}{b} = \frac{c}{d}$, 12. 11. $\frac{4}{5}$.
 12. Lengths are 6 and 14 in.; volumes, 95.4 c. in. and 222.7 c. in.
 13. $\frac{2}{3}$. 14. 325.

Ex. 26

1. $\frac{5}{4}$ in each case. They are equal. Direct proportion. 2. $\frac{3}{4}$ in each case. Direct proportion. 3. $\frac{3}{8}$ and $\frac{3}{8}$. The second ratio must be inverted.
 4. $\frac{150}{90} = \frac{5}{3}$. 5. $\frac{5}{7}$ and $\frac{3.5}{2.5} \left(\frac{7}{5} \right)$. Inverse proportion, because the second ratio has to be inverted to produce the equality of the ratios. This is logical, since to maintain equal areas it is obvious that if the base is *increased* from 5 to 7 in. the altitudes must be *correspondingly decreased* from 3.5 to 2.5.
 6. $\frac{4}{2.5}$ or $\frac{8}{5}$; $\frac{16}{6.25}$ or $\frac{64}{25}$; $\frac{8}{5}$ needs to be squared.
 7. Ratio of times = $\frac{3}{2}$. Ratio of distances, $\frac{6}{14.4}$, or $\frac{5}{6}$. Comparison, $\frac{3}{2} = \sqrt{\frac{5}{6}}$. 8. $\frac{4}{5}$ or 2; $\frac{8}{16}$ or $\frac{1}{2}$. 2 is the reciprocal of $\frac{1}{2}$. The arms are inversely proportional to the weights. 9. $\frac{1}{16}$ and $\frac{9}{16}$. None apparent.

Ex. 27

1. 7.996, or 8 approx. 2. $\frac{x}{y} = \frac{a}{b}$. 283.8 m. 3. 4.07, $\frac{c^2 d^2}{b^2}$, 2.5.
 4. £5, 11s. 3 $\frac{3}{4}$ d. 5. 13s. 4d. 6. £19. 7. £4, 16s. 7d.; £2, 8s. 3 $\frac{1}{2}$ d.
 8. $\frac{3x^2}{16}$, 300. 9. $\frac{40}{11}$, 42.5 oranges. 10. $\frac{b}{x}$, 6 h. 11. £5, 3s. 5 $\frac{1}{4}$ d.
 12. £21, 19s. 1 $\frac{1}{4}$ d.

Ex. 28

1. Because the sides of the similar triangles similarly placed about the equal angles are proportional. 2. 1.6 and 2.35. 3. 3.48 in.
 4. 13.05 cm. 5. (a) 6, 5.2, 3; (b) 8, 6.9. 6. 3, 1.5, 2.6; 5, 2.5, and 4.3.

Ex. 29

1. 45s. 2. £175. 3. £3, 2s. approx. 4. 3.8 lb. 5. 40 cows.
 6. 11 oz.

Ex. 30

1. 1s., 18s., 7s. 6d., £173, 15s. Note 5% = 1s. in £1. 2. £1, £6, 10 guineas, £23, 6s. 8d., 3380 m., 80 litres, 140 oz. or 8 lb. 12 oz. 3. 6d., 3s. 10½d., 21s. 9d., 1s. 8d. Note 2½% = 6d. in £1. 4. £5, 9s.
 5. £18, 8s. 2½d. 6. 24. 7. £31, 18s. 7½d.; £127, 14s. 6d.
 8. 1572.5 c. in. copper; 155.5 c. in. tin. 9. 33⅓%. 10. £133, 6s. 8d.
 11. He gains by 14.1%. 12. £1000. 13. 94.1% and 5.9%.
 14. 5.7%. 15. £242 approx., £1052. 16. £21.84; £3.16 gain.
 17. 32.0128 ft. 18. £180, £112.5, £157.5.

Ex. 31

1. £35. 2. £19, 18s. 1½d. 3. £4, 7s. 4. 4%. 5. 1.176%.
 6. 2½ yr. 7. 10⅝ yr. 8. 4%. 9. 2.1 yr. 10. £149, 14s. 3⅓d.
 11. 40 yr. 12. £505, 11s. 1½d. 13. £274, 7s. 9½d. approx.
 14. £568⅓. 15. 4⅓%.

Ex. 32

1. £250, £160, £800; £225, £144, £720. 2. £165. 3. £938, 11s. 9d.
 4. £18. 5. £165. 6. 90. 7. 129. 8. 5000.
 9. £6431, 5s. 10. £4070, 9s. 3½d. 11. £3557, 11s. 6d. 12. £4165.
 13. £400 stock; £10.

Ex. 33

1. £24, 8s. 2. £35, 19s. 9½d. 3. £20, 5s. 4. £15, 2s. 6d.
 5. From (a) £20½⅞ and from (b) £19⅓⅞. The former. 6. £7, 5s. 10d.
 + £6, 15s. 7d. + £6, 11s. 4d. = £20, 12s. 9d. 7. The latter.
 8. £4, 15s. 2½d. 9. 4½⅞%. 10. The former.

Ex. 34

1. 93½, 95½, 120¼, 58, 133¾, 100½. 2. 93½, 95½, 120, 57¾, 133½, 99⅝.
 3. Income changes from £14.28 to £17.62. ∴ An improvement of £3.34.
 4. £1031, 5s. 5. £1370, 5s. 6. £22, 10s.; £486, 5s.
 7. £547, 16s. approx. 8. 4⅞%. 9. £335, 11s. 3d.
 10. £636, 13s. 3d. 11. £6032½.

Ex. 35

2. Parallelogram or rhomboid. 3. (11·5, 15). 4. 60, 120, 150, 300 gr.
 5. 4·8, 3·2, 34 cm.; 50, 37·5, 30, 20 cm. 6. 10s., 15s., 8s., 10 francs,
 9 fr. 375, 20 francs, 18 fr. 75. 7. 1·1, ·6, 1·75 lb. approx. 8. 14s.,
 27s., 30s. 9. £6, £4, 10s., £4, 1s., £2, 16s. 8d. 10. 4s. 6d., 4s. 8d.

Ex. 36

5. 1350, 1700. 6. 14,770, 14,350, 13,490. 7. 298. 8. 9s. 6d.,
 3s. 6d.

Ex. 37

1. 2·5 d., $\frac{1}{2}$, $\frac{2}{3}$, $\frac{1}{2}$ or $\frac{1}{2}\frac{0}{8}$, $\frac{5}{24}$, $\frac{1}{2}$. 2. 6 in. 3. On the 4-in. mark.
 4. $V \propto \frac{1}{p}$. (a) 5 c. in. (b) 20 c. in. 5. $C \propto \frac{1}{R}$, 1 amp., 10 amp.
 6. 1 h. 7. 5 times the distance. 8. 4.

Ex. 38

1. 45 men. 2. $3\frac{1}{8}$ d. 3. 40 d. 4. $286\frac{2}{3}$ m. 5. 22·5 min.
 6. 4·94 in. 7. 7·24 cm. 8. 11·56 in. 9. $6\frac{2}{3}$, 8, $13\frac{1}{3}$ lb.
 10. 1·45 lb. 11. 17·4 in. 12. $\frac{1}{2}$ oz. 13. 21·5 amp. 14. 3·26 c.c.,
 10·86 c.c., 32·6 c.c. 15. 6·6 c. in. 16. 180 lb.; 360 lb. to sq. in.
 17. 216 lb. per sq. in. 18. $177\frac{7}{8}$ lb. per sq. in.

Ex. 39

1. (a) 10·3; (b) £·964 or 19s. $3\frac{1}{4}$ d.; (c) 14·28; (d) $1·492 \times 10$.
 2. (a) $42\cdot\dot{6}$; (b) 5·0001; (c) 11·45; (d) 27 yd. 3. (a) 25; (b) 22·4;
 (c) 15·74; (d) 6 in.

QUESTIONS ON METHODS OF MEASURING ANGLES

1. No. Being a radius of the circle, and all radii of a circle being equal, it remains of constant length. 2. They increase. Yes. 3. It increases in length as the angle increases, and vice versa (direct variation). 4. It increases as the angle decreases (inverse variation). 5. R approaches A, and the tangent shortens. 6. Obtuse. 7. θ is bisected. Each equal angle = $\frac{\theta}{2}$. 8. Nearer A. θ is smaller. 9. θ must be increased.

Ex. 40

1. $\frac{2}{3}$, $\frac{1}{2}$, $\frac{5}{18}$, $\frac{1}{4}$ or ·25, $\frac{9}{20}$ or ·45, ·01 or $\frac{1}{100}$, $\frac{1}{500}$ or ·002, $\frac{9}{2500}$. 2. 180°, 120°, 36°. 3. 11·25° or 11° 15', 7·5° or 7° 30'. 4. (a) 22·8'; (b) 15'; (c) 6'; (d) 135'; (e) 243·6'. 5. 16° 5' 25". 6. 45·251°. 7. 200, 160, 287·5. 8. (a) 50, (b) 20, (c) 125. 9. 0° 36' 57"; 462° 83'; 3'; 62° 38' 7". 10. 50°, 22·2°, 83½°. 11. 12·24°, 51·66°, 30°. 12. 19½° or 19° 14' 31·4". 13. 45° by ·5°.

Ex. 41

1. 1.04 in. and 3 in. 2. 50° . 3. $.775, 46^\circ$. 4. 98° and 1.5 approx.
5. 1.5155 in. 6. $.845, 50^\circ$. 7. (a) 7.264 in.; (b) 9.08 in.; (c) 6.356 cm.
8. 90° . 9. 60° and 180° . 10. 2.3 in. 1.732.

Ex. 42

(a) 2π times, or 6.2832 times. (b) No. The chord of $60^\circ = 1$, and the chord is shorter than the arc. Thus 1 radian will be a little less than 60° . It is 57.3° . (c) 180° . (d) 90° . (e) $\frac{1}{\pi} = \frac{x^\circ}{180}$. $\therefore x^\circ = \frac{180}{3.1416} = 57.3^\circ$. (f) $360^\circ, 2\pi$. (g) 2π times, or $2 \times 3.1416 \times 2 \text{ ft.} = 12.5664 \text{ ft.}$

Ex. 43

1. 1.0472 cm., 6.2832 cm., 5.86432 in. 2. π or 3.1416 radians
- $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{6}$; or 1.5708, .7854, 1.0472, .5236. 3. (a) 1.309 radians; (b) 75° .
4. 3.6 in. approx. 5. 3.1416 and 5.4978 cm. $\frac{4}{3}$. 6. .15708, .7854, .1745, .0175, .0698.
7. .5236 in., .7989 in., 1.0652 in., 1.3315 in., 1.5978 in. Arc = $r\theta$.

Ex. 44

1. 57.29° . 2. $60^\circ, .1745$. 3. 8 revs. 4. $2\pi, 4\pi, 100\pi, 6.2832 \text{ m.,}$
 $12.5664 \text{ m., and } 314.16 \text{ m.}$ 5. (a) 480π radians; (b) $480 \times 3.1416 \text{ ft.,}$
or 1507.9 ft.; (c) 4523.9 ft.; (d) 2261.9 ft. per minute. 6. $\frac{3}{4\pi} \text{ sec.,}$
or .238 sec.; $\frac{4\pi}{3}$, or 4.1888 radians per second. 7. 11.06 sq. cm.
8. 52.5 sq. in. 9. 3.2725 sq. in. (sector) - 2.7063 (triangle) =
.5662 (area of segment). 10. Each sector is 5.53 sq. in. and each seg-
ment 0.95 sq. in. 11. 72 sq. in. 12. 75.3984 sq. in., or 75.4 approx.
13. 2 in. 14. 30 in., 4.45 radians. 15. (a) 243.28; (b) 4865.7; and
(c) 5108.9 sq. cm.

ANSWERS TO QUESTIONS ON FIG. 31.—(a) Increases. (b) Decreases. (c) Perp. decreases; base increases. (d) No. (e) $\frac{3}{4}, 12\frac{1}{2}''$, because they are similar triangles. (f) Unchanged. (g) Yes. (h) $\frac{4}{5}$. (k) Made less.

Ex. 45

1. .5 in., .75 in., 1 in., 1.25 in., and 1.5 in., .5, $\sin 30^\circ = .5, .5 \text{ in.,}$
.5 cm., .5 m. 2. .85 in., 1.275 in., 1.7 in., 2.125 in., 2.55 in. approxi-
mately, $\sin 60^\circ = .85 \text{ approx.} = .866$. 3. 2 in., 4 in., and 5 in.; ratio
in each case is $\frac{1}{2}$, or .5; $\cos 60^\circ = .5$. 4. 250 ft. 5. $\sin 45^\circ = \cos 45^\circ$
= .7071. 6. The ratio of its perpendicular to the hypotenuse.
7. The ratio of the base to the hypotenuse; .95 approx., or .966 = $\cos 15^\circ$.
8. 48.29 ft. high, 12.94 ft. from foot of wall. 9. 4.33 in.

Ex. 46

1. .7 ft. or 1 ft. 2. 2.598 in. (2.6 approx.). 3. 2.8 cm. approx. and 24.4 sq. cm. 4. 2.8 cm., 18.2 sq. cm. 6. 9.397, 7.66, 6.428, and 1.736 cm. respectively. 7. 4 in., 2 in., 1.5 cm., and 6.9 in., 3.46 in., 2.6 cm. approx. 8. 8.26 in. 9. 74.16 sq. cm. 10. 13.24 cm., 25.5 cm., 28.1 cm. 11. 53° approx. .8. 12. 30°. 13. 6.08 in. 14. 38°, 51°, 91°.

Ex. 47

A.—1. 180°, 2 right angles, π radians. 2. 80°. 3. 180°, 2 right angles. They equal the three angles of the triangle ABC. The two interior angles opposite the exterior angle. 4. 90°. No; every angle at C is a right angle. 5. 90°. No. 60°, 30°. Because the addition of one to the other makes 90°—one is the “complement” of the other.

B.—1. 5 cm., 4 cm. 2. 75°. The longest side is opposite the largest angle, and the shortest side is opposite the smallest angle. 3. Because it is opposite an obtuse angle. Because BD is the shortest side of the triangle. The angles A and B together equal the exterior angle D.

Ex. 48

1. Two. 5. The perpendiculars are equal because they subtend angles of 15° in equal triangles. The arms are tangents because they are at right angles to the perpendiculars which have become the radii of the circle. 6. All sides of the triangle become tangents to the circle since they are equidistant from O.

Ex. 49

1. The triangles ADC and CDB are similar. Thus AD bears the same ratio to DC as DC does to DB, &c. 2. ADC, CDB, and ACB. 3. The angle DCB. 4. The triangles ADC and CDB are similar, and these are the sides about the equal angles. 5. CD is the mean proportional of AD and DB. 6. (b) $\frac{2}{3.46} = \frac{3.46}{6}$. They are equal. (c) Make a line (2 + 6) cm. long. On it construct a semicircle, and from the junction of 2 cm. and 6 cm. raise a perpendicular to meet the semicircumference. The length of this perpendicular is the mean proportional between 2 and 6. (d) 3.46 in. 7. 4. 8. 4 cm. The side of the square is the mean proportional between the sides of the rectangle. 9. Fig. 36 with AD 4 and DB 9 units of length, then CD is 6 units of length.

Ex. 50

1. 2 in. $\sin A = \frac{2}{2.5} = .8$. $\cos A = \frac{1.5}{2.5} = .6$. $\tan A = 1\frac{1}{2}$.
 $\sin B = .6$. $\cos B = .8$. $\tan B = .75$. 2. About 30.5°.

$$\tan A = \frac{1}{1.7} = .588.$$

3. That in a 60° right-angled triangle the perpendicular is 1.732 times the base. $1\frac{1}{2}$ in. 4. About 64° . 70 ft.
 5. 47.7 ft. approximately. 6. 148.5 ft. 7. 62.49 yd. 8. 53° , 37° ,
 17.34 sq. cm. 9. 24 m. 10. .424.

Ex. 51

1. (i) 24° , 45° , 65° ; (ii) 55° , 20° , 54° . 2. 40° , 25° , 46° , 50° .
 3. .5736, .8192, .7002. (a) 5.736 cm. and 8.192 cm.; (b) 4.0152 cm. and
 5.7344 cm. 5. 1. 6. 1.7321. $\tan 60^\circ$. 7. 17.93.
 8. 13.44 sq. cm. 9. 19.446. 10. Other angle is 53° . Sides are
 5.32 cm., 6.6 cm., and 4 cm., approximately 10.66 sq. cm. 11. 2198 ft.
 approximately.

Ex. 52

1. +5.1, +3.1, -0.1, +1.5, +1.2, +0.9. 2. 105° , 15° , 240° , and 120° .
 3. 21 m.; $1\frac{1}{7}$ h. 4. 70 m. and 10 m. per hour. 5. 55 m. per hour.
 6. 600 knots, 360 knots. 7. A square each side of which is about 11.1
 squares long; area 123 squares approximately. 8. 105 sq. in.; 15 in.
 9. 153.94. 10 and 11. 0, 3. 12. +4 units.

Ex. 53

1. (i) $6a$; (ii) 27 l.; (iii) 4 m. 2. (i) $7a + 3b$; (ii) $6a^2 - 9ab$;
 (iii) $3a + 2b - c$. 3. 13, -12, 5.5. 4. $11a - 11b + 4c$. 5. 97.
 6. $x^3 + 4x^2y - 20xy^2 + 14xy^3 + 14y^4$, 541. 7. +9a, +19a, -2a, 0.
 108 in., 228 in., -24 in., 0 in. 8. $a^2 - 9ab$. 9. +b + 11c. 10. 6xy.
 11. (a) 6; (b) +14; (c) +24; (d) -18; (e) -6; (f) +b; (g) +c; (h) +b;
 (k) +12a + 3b - 3c; (m) 2a + 3b - 6c. 12. (a) -14; (b) +14;
 (c) a + 4c; (d) 3a - 5b; (e) -3a + 2b; (f) -9a - b + c. 13. (a) 49;
 (b) 1; (c) 4500; (d) 0; (e) 32; (f) $\frac{2}{3}$; (g) $\neq 7$.

Ex. 54

1. (a) $6a - 3a - 2a = a$; (b) 9a; (c) 5a. 2. (a) 45; (b) $21\frac{3}{8}$;
 (c) $-3a + 5b - 3c$. 3. (a) $.78a^2 + 20.585ab$; (b) $2b + 2c - 2a$.
 4. $6x - 9y + 16z$. 5. 5x. 6. (a) $-1\frac{1}{2}1$; (b) -64; (c) 76. 7. 16.

Ex. 55

1. 12.5664, 10.44, .354 sq. ft. 2. 1443 sq. ft. 3. 25.1328 sq. in.
 4. 10.603 sq. in., 43.397 or 43.4 c. in., 1.1283 lb. 5. 99.9 sq. ft. or
 100 sq. ft. approximately. 6. 87.3175 sq. ft.

Ex. 56

1. (a) 2; (b) -1.06; (c) 10; (d) 2; (e) 4; (f) 2. 2. $(49 + x)$ or
 $(x + 49)$. The numbers are 2 and 51. 3. 2. 4. (a) 3;
 (b) 3.5; (c) 1.25. 5. (a) $a + b + c$; (b) $c - (a + b)$ or $c - a - b$.

6. (a) $\sqrt[3]{a^2b}$; (b) $a^2 + \sqrt[3]{b}$; (c) $a^2 + b^2$; (d) $\sqrt{a-b-c}$. 7. 6 and 15.
 8. 18. 9. $(x+1)$, $(x-1)$. 21. Equation is $x + (x+1) + (x-1) = 63$.
 10. 3x. Length, 900 yd.; breadth, 300 yd.; area, 270,000 sq. yd.
 11. 10. 12. 12 m.

Ex. 57

1. Hypotenuse is 5 in. The sum of the areas of the squares on the two sides and the square on the hypotenuse are each 25 sq. in. 2. Your hypotenuse will be 13 cm. Again your results are equal. 3. Yes, the sum of the squares on the two sides equals the square on the hypotenuse in area. 4. By demonstration. 5. The areas of the squares on the hypotenuse and the two sides will be a^2 , b^2 , and c^2 . Since the first is equal to the sum of the other two, therefore $a^2 = b^2 + c^2$.

Ex. 58

1. 4·8 in. 2. 7·72 in. 3. 39·6 ft. 4. 15·01 in. 5. 603·6 ft.
 6. 705·9 ft. 7. 217·2 ft. 8. 7·38 ft. 9. 72·2 knots.
 10. 2089·2 sq. ft., 7661 tiles. 11. 21 ft. 12. 6·6 in. 13. 4·7 cm.
 14. 10·2 cm.

Ex. 59

1. 4999. 2. 326·8. 3. 6·741. 4. 2886 m.
 5. 4·326. 6. 1149 c.c., 45·5 Kg. 7. 33·6 in.
 8. $\frac{2x}{m}$. 9. (a) $6a(mx^5 - ny^2)$; (b) $\frac{(a+b)^2}{xy^2}$. 10. 116·9 sq. cm.

Ex. 60

1. 184·96, 2515·4, 3·68. 2. (a) a^{10} ; (b) x^{15} ; (c) $y^{\frac{3}{2}}$; (d) m^{-9} ; (e) $l^{3\cdot1}$;
 (f) $r^{2\frac{2}{3}}$; (g) 7^5 ; (h) $10^{1\cdot62903}$. 3. (a) x^3 ; (b) $a^{2\cdot5}$; (c) $a^{-1\frac{1}{2}}$; (d) s^{12} ; (e) 10;
 (f) $t^{5\cdot64}$; (g) $y^{2\cdot41}$; (h) $m^{1\frac{1}{2}}$. 4. (a) $24a^3b^4c$; (b) $48c^2xyz$; (c) $16m^2a^4x^3y$;
 (d) $3al^2s^2t^3$; (e) $7abc$; (f) $7x^2b^3$; (g) $8\cdot5bc$; (h) $\cdot125x^4y^4$. 5. $272a^4b^3c^4$.
 6. $-2x^2y$. 7. $15\cdot12abx^3$. 8. $-4nx$. 9. $x^2y = 2\frac{1}{4}$, $y = 4\frac{1}{2}$.
 10. $3\frac{1}{2}$ in.

Ex. 61

1. (a) $-3x^7 - 4x^6y - 8x^5y^2 + 6x^3y^4 + 12x^2y^5 + 7xy^6 - 6y^7$. (b) $-m^6 + m^5 + 2m^4 + 7m^3 - 8m^2 - 4m - 6$. 2. (a) $18x^2 - 24xy + 36y^2$;
 (b) $9x^3 - 12x^2y + 18xy^2$; (c) $-9x^3 + 12x^2y - 18xy^2$; (d) $12x^2y - 16xy^2 + 24y^3$;
 (e) $-21x^2y + 28xy^2 - 42y^3$; (f) $9ax^2 - 12axy + 18ay^2$; (g) $-21x^4y^2 + 28x^3y^3 - 42x^2y^4$; (h) $-24xz + 32xyz - 48y^2z$. 3. $-42p^6 + 18p^5 - 12p^4 + 6p^3 - 48p^2$. 4. $48m^2 + 24md$. 5040 sq. dm. or 50·4 sq. m.
 5. $a^4 + 2a^3 - 4a^2 + 8a - 7$. 6. (a) $-7x^5 + 12x^4 - x^3 + 6x^2 + 4x - 8$;
 (b) $a^3b - 2a^2b^2 - 2ab^2 + b^4$; (c) $a^3 + a^2b - ab^2 - b^3$. 7. $a^2 - b^2 - c^2 - 2bc$.
 8. (a) $a^2 + 2ab + b^2$; (b) $a^2 - 2ab + b^2$; (c) $a^2 - b^2$; (d) $a^4 - b^4$; (e) $a^3 + b^3$;
 (f) $a^3 - b^3$. 9. $12ax - 9bx + 6cx$. 10. (a) $12xy^2 - 8y^2$;
 (b) $-3x^4 + 2x^3y - 4x^2y^2$.

Ex. 62

1. 7×7 , 7×3 , 3×17 , 7×17 , $a \times b$, $7 \times b \times c$, $13 \times m \times m$, $7 \times 2 \times m \times n$.
 2. (a) 2, 64; 4, 32; 8, 16. (b) 16, a^2b ; $2a$, $8ab$; $4a^2$, $4b$, &c. (c) $5a^3$, $5b^2$; 25 , a^3b^2 , $25a^3b$, b^2 , &c.; (d) 2, 52; 4, 26, 8, 13. 3. (a) 3 and $(a - b)$; (b) 7 and $(m + n)$; (c) a and $(x^2 - y^2)$; (d) x^2 and $(y + z)$. 4. $21 = 3 \times 7$.
 $27 = 9 \times 3$. 3. 5. Common factor $3ab$. 6. $x^2 + 5x + 6$, $x^2 + 13x + 42$, $x^2 + 8x + 15$. (i) By multiplication of numerical terms. (ii) By addition. 7. (a) $x^2 + 9x + 14$; (b) $x^2 + 12x + 32$; (c) $(x + 3)(x + 2)$; (d) $(x + 5)(x + 2)$. 8. (a) $x^2 - 5x + 6$; (b) $x^2 - 13x + 42$; (c) $x^2 - 8x + 15$.
 (i) By multiplication. (ii) In adding the two part products two minus quantities have to be combined. (iii) Two minus quantities multiplied give a plus quantity, but added give a minus quantity. 9. $x^2 + 10x + 21$; $x^2 - 10x + 21$; $x^2 - 11x + 18$; $x^2 - 12x + 32$; $(x - 4)(x - 3)$; $(x - 5)(x - 2)$.
 10. 9, 25, $9m^2$, $x^2 + 4x + 4$, $x^2 + 6x + 9$, $x^2 - 16x + 64$, $x^2 - 2x + 1$.
 11. $x^2 - 1$, $x^2 - 4$, $x^2 - 9$, $x^2 - 25$, $x^2 - 36$, $x^2 - 49$. 12. $x^2 - x - 20$, $x^2 - 5x - 24$, $x^2 + 3x - 28$, $x^2 + 7x - 18$. Observations: (i) Last term is product of the numbers in each case. (ii) Coefficient of x is the difference of the numbers. (iii) Sign of last term is minus because obtained by product of a plus and a minus quantity. (iv) Sign of middle terms depends on sign of the larger number in the factors.

Ex. 63

1. $(x + 3)(x + 1)$. 2. $(x - 4)(x - 3)$. 3. $(x - 6)(x - 4)$.
 4. $(x + 3)(x + 5)$. 5. $(x - 5)(x - 4)$. 6. $(x - 13)(x - 3)$.
 7. $(x - 4y)(x - 3y)$. 8. $(m + 5n)(m + 3n)$. 9. $(pq - 1)^2$.
 10. $(bc - 3)(bc - 2)$. 11. $(a - 100)(a - 2)$. 12. $(x - 15y)(x - 4y)$.

Ex. 64

1. $(x - 3)(x + 1)$. 2. $(x - 6)(x + 5)$. 3. $(x - 40)(x + 5)$.
 4. $(x + 7)(x - 4)$. 5. $(x + 6)(x - 1)$. 6. $(a - 12b)(a + 7b)$.
 7. $(m - 5n)(m + 2n)$. 8. $(xy + 5)(xy - 4)$. 9. $(b + 15)(b - 6)$.
 10. $(p + 21)(p - 6)$. 11. $(m - 17)(m + 13)$. 12. $(x - 60)(x + 6)$.

Ex. 65

1. $x^2 + 16x + 64$. 2. $x^2 - 10x + 25$. 3. $x^2 + 4xy + 4y^2$.
 4. $x^2 - 14xy + 49y^2$. 5. $36 + 12x + x^2$. 6. $81 - 18y + y^2$.
 7. $m^2 + 8mn + 16n^2$. 8. $p^2 - 22pq + 121q^2$. 9. $(x - 5)(x + 5)$.
 10. $(l - 3t)(l + 3t)$. 11. $(R + r)(R - r)$. 12. $\pi(R - r)(R + r)$.
 13. $(4 - m)(4 + m)$. 14. $x(x + 2y)$. 15. $2y \times 2x = 4xy$.
 16. $(x^2 + y^2)(x + y)(x - y)$. 17. $(5x - 7)(5x + 7)$. 18. $(25m - 1)(25m + 1)$.
 19. $(4x^2 + 9y^2)(2x - 3y)(2x + 3y)$. 20. $(2 - x + y)(2 + x - y)$.
 21. $(6 \cdot 3 - 2 \cdot 7)(6 \cdot 3 + 2 \cdot 7) = 32 \cdot 4$. 22. $(146 \cdot 8 - 3 \cdot 2)(146 \cdot 8 + 3 \cdot 2) = 21540$.
 23. $(1 \cdot 385 - \cdot 615)(1 \cdot 385 + \cdot 615) = 1 \cdot 54$.
 24. $(2 \cdot 6x - 5 \cdot 4y)(2 \cdot 6x + 5 \cdot 4y)$.

Ex. 66

1. $(3x + 4)(2x - 3)$.
2. $(3x - 4)(2x - 3)$.
3. $(5x - 8)(x - 6)$.
4. $(2x + 5)(4x - 4)$.
5. $(4x + 3)(2x + 5)$.
6. $(7m + 2n)(2m - 5n)$.
7. $(3x - 5)(2x - 3)$.
8. $(2x + 5)^2$.
9. $(3x + 4y)^2$.
10. $(4x - 5)(x - 5)$.
11. $(3x + 2y)^2$.
12. $(16xy - 5)^2$.

Ex. 67

1. $2ab(2a - 3b)$.
2. $ab(a^2 - ab + b^2)$.
3. $3x^2(x - 2)$.
4. $a(a + ax + bx)$.
5. $48a^2bc(2ab - c^2)$.
6. $5ab(4x - 3a + 5a^2b)$.
7. $(a + b)(c - d)$.
8. $(2a^2 + 6)(a - 2)$.
9. $(xy + z)(xy + 2)$.
10. $(m^2 - 1)(7m - 3)$ or $(m + 1)(m - 1)(7m - 3)$.
11. $(p^2 - r^2)(p + r)$ or $(p - r)(p + r)(p + r)(p + r)$.
12. $(a - b)(c - d)(c + d)$.

Ex. 68

1. $a(bc - 3)(bc - 2)$.
2. $a(b + 15)(b - 6)$.
3. $\pi(R - r)(R + r)$.
4. $y(2x - y)$.
5. $(a + b - c)(a + b + c)$.
6. $(3x + 4y - 2z)(3x + 4y + 2z)$.
7. $(-2m)(2a) = -4am$.
8. $ab(x - 3)(x + 3)$.
9. $m(2r - 4s)(2r + 4s)$ or $4m(r - 2s)(r + 2s)$.
10. $(q - r - s)(q + r + s)$.
11. $3y(x - 3y)(x + y)$.
12. $16x(2x + 5)(x - 1)$.

Ex. 69

1. 3.
2. $10 \cdot 05$.
3. 7.
4. 7.
5. 4.
6. 7.
7. 8.
8. 4.
9. -5.
10. $-11 \cdot 9$.
11. $4\frac{1}{2}$.
12. $-2\frac{1}{18}$.
13. 3.
14. $-3 \cdot 5$.

Ex. 70

1. 72 and 56.
2. 20, 4.
3. 4.
4. 27.
5. 24 and 25.
6. 2000 and 600 yd.
7. $\frac{4}{3}$, 2 and $4 \cdot 5$.
8. $A = 238 \cdot 7$ sq. in.
9. $\text{£}64\frac{2}{7}$.
10. 8 apples.
11. 6 men.
12. 15s.

Ex. 71

1. $65 \cdot 97$ sq. cm.
2. $43 \cdot 98$ sq. cm.
3. $36 \cdot 7$ sq. in.
4. 76 sq. yd.
5. 68 sq. in.
6. $29 \cdot 85$ sq. ft., $127 \cdot 25$ sq. ft.
7. $10 \cdot 4$ c.c., $92 \cdot 59$ g.
8. 6 in.
9. $8 \cdot 71$ yd.
10. $2 \cdot 52$ in.
11. $6 \cdot 4$.
12. $14 \cdot 5$ cm.

Ex. 72

1. $\frac{4}{3}x^3y^4z^3, -2$.
2. $-4m + 2m^3 + 3m^6$.
3. (a) $(a + 2)$; (b) $(5a - 3b)$; (c) $(x + 6)$; (d) $(x - 2)$.
4. (a) 1; (b) $\frac{1}{2}(s - 3)$.
5. $(x - 2)$.
6. $3x^4 - 9x^3 + 8x^2 - x - 1$.
7. $a^2 - 2ab + b^2$.
8. $a^3 + 3a^2b + 3ab^2 + b^3$.
9. $26 + 13a + 8a^2 + 4a^3 + 2a^4 + a^5$.
10. $x^2 - 2xy + y^2 - 2x - 2y + 1$.

Ex. 73

1. $552 \cdot 7$ c. in.
2. $197 \cdot 4$ c. in.
3. $753 \cdot 98$ sq. in.
4. $10 \cdot 8$ c.c.
5. $60 \cdot 5$ in.

Ex. 74

1. 284·6 c.c. 2. 22·4 in., 7367 c. in. 3. 2143 c. in. 4. 5 in.
 5. 153·9 sq. in., 179·6 c. in. 6. 9 in. 7. 12·41 lb. 8. 50·27 sq. in.,
 113·1 sq. in., 125·7 sq. in. 9. 10 in., 261·8 c. in., 56·55 c. in., ∴ vol.
 of frustum = 205·25 c. in. 10. 101·4 c.c., 16·9 c.c. 11. 523·6 c. in.,
 261·8 c. in. 12. $\frac{1}{2}$ or $\frac{2}{3}$. 13. 904·8 c. in., 268·1 c. in., $\frac{27}{8}$.
 14. 416·95 c. in.

Ex. 75

1. 70° . $b \sin A = a \sin B = \text{length of CD}$. 2. Yes. 3. 64° ap-
 proximately and 50° approximately, ·89 and ·766. Methods are: (a)
 $PQ = \sqrt{3^2 + 6^2}$ and (b) $PQ \sin 64^\circ = 6$ or $PQ = \frac{6}{\sin 64^\circ}$. $PQ = 6·7$ cm.,
 $QR = 7·8$ cm. 4. (a) 6 in.; (b) 6 in.; (c) the perpendicular QX;
 (d) Yes. PQ , $\sin P$, and $QR \sin R$ each equal the perpendicular. Thus the
 sides of a triangle are proportional to the sines of the angles they subtend.
 5. 12·5 cm. $\sin C = \cos B = \cdot 8$ and $\cos C = \sin B = \cdot 6$. 6. AD
 = 3·464 in., DC is 2 in. $\sin A = \cdot 5 = \cos C$; $\sin C = \cdot 866 = \cos A$;
 $\tan C = 1·732 = \frac{1}{\tan A}$.

Ex. 76

1. $\frac{·6428}{·7071}$ or ·909. 9·09 in. 2. 90° . 3. By substitution.
 4. $30^\circ, 60^\circ, 90^\circ$. 5. $\sin^2 \theta = 1 - \cos^2 \theta$ and $1 - \cos^2 \theta = (1 - \cos \theta)(1 + \cos \theta)$;
 $\sin \theta = \cdot 6$. 6. 8·74 in. 7. 2·68 in. 8. $80^\circ, 9·02, 9·45$.

Ex. 77

1. (a) 1·05, 1·58, 2·25; (b) $\frac{3}{2}$ or 1·5; (c) $\frac{2}{3}$ or $\frac{1}{2}$; (d) same as (c).
 2. (a) $\frac{3·5}{·5}$ or 7; (b) $\frac{·5}{1}$ or 2; (c) $\frac{7}{1}$ and $\frac{5}{3}$. 3. (a) $\frac{3}{5}$; (b) 7s. 6d. and 12s. 6d.

Ex. 78

1. 2 in. and 3 in. 2. 3·6, 5·1, and 1·3 cm. 3. £3, 6s. 8d.; £6, 13s. 4d.
 4. 90, 120, 150. 5. £16, 13s. 4d.; £3, 6s. 8d.; £20. 6. £39, 7s. 6d.;
 £53, 3s. 1½d.; £110, 4s. 4½d.; 5s. in the £1. 7. 12 of each.
 8. $3\frac{1}{2}$ and $4\frac{2}{3}$ in. 9. The smaller weight is $1\frac{2}{3}$ the distance of the
 larger. 1 cm. 10. £141, 13s. 4d.; £91, 13s. 4d.; and £66, 13s. 4d.

Ex. 79

1. 1·732, 2·449, 3·464, 4·899, 5·197, 2·645, 5·385, 12·12. 2. (a) 28·54;
 (b) $5\sqrt{7}$ or 13·225; (c) 273; (d) 21·5. 3. 32·908 or $19\sqrt{3}$. 4. (a) 168;
 (b) 48; (c) $18\sqrt{5} + 75$; (d) -270; (e) 61. 5. 8, 12 - $6\sqrt{3}$ or 1·608,
 11 - $4\sqrt{7}$, 9. 6. (a) 5·66; (b) 8·66; (c) 1·577; (d) ·452; (e) 8·882.
 7. 2·248.

EX. 80

1. 43·2. 2. 3·4 days. 3. Constant = 1. 4. 46·8 sq. cm.
 5. 113·1 c. in. 6. 28·4. 7. 10·8 lb. 8. 18 ohms.
 9. 10 sec. 1024 ft. 10. 16·24 c.c.

CONCLUDING EXAMPLES

- A.—1. 555·2, 3·873. 2. 25·7916. 3. ·8290, ·9511, ·9325, 60°, 45°, 45°.
 4. $\frac{10c^3 - 5(a^2 + b^2)}{3a(b - c)}$. 5. 1·02.

- B.—1. 4·786, 7·091 m. 2. 6·109 sq. m. 3. 17·4. 4. 44·178 Kg.
 5. 565·5 c. in.

- C.—1. 5·04 in. 2. 21·9 ft. 3. 4·72 shares. 4. 5%. 5. 5%.

- D.—1. 6·931. 2. 3s. 6d. 3. $(x - 4)(x - 3)$; $(3a + 7p)(2a - 5p)$;
 $3(3a - 2b)(3a + 2b)$. 4. $2x^6 + x^5 - 6x^4 - 6x^3 + x^2 + 15x - 5$.
 5. 108·8 ft.

- E.—1. 10 ft. 5 in. and 4 ft. 2 in. 2. 4·4305. 3. 9; 59,049.
 4. Use a square. 5. $3·142 \times 6·84 = 21·48$. $\frac{14680}{4·301} = 3413$.

- F.—1. 2·54 cm., 39·37 m. 2. 6·452. 3. ·25%. 4. 1·844 Kg.,
 3·68 Kg. 5. 1·88 cm.

- G.—1. $\sqrt{7}$ is to $\sqrt{5}$. 2. 23·83. 3. 13·16. 4. 38 cm.
 5. 63·21 sq. in.

- H.—1. 476·2 revolutions. 2. 11433·3 g. 3. 10 cm.
 4. $3a + 5b - \frac{8\sqrt{a}}{(a + b)^3}$. 29144 $\frac{1}{11}$. 5. £5, 19s.

- I.—1. £52, 10s. 2. 17·3%. 3. £11,404, 2s. 6d. 4. 16·04 m.
 5. 40 Kg. per sq. cm.

- J.—1. 9·6. 2. 11·56 and 6·125. 3. 173·1 sq. in.
 4. By substituting values. 5. 26·5 cm.

- K.—1. $6abc + c^3 - b^3 - a^3 + 6b^2c - 6bc^2 - 6ab^2$. 2. $7x^2 - 6xy - 8y^2$.
 3. ·2 and $-7\frac{5}{13}$. 4. $x^2 - 6x + 9$. 5. 153·5.

- L.—1. 1885 c. in. 2. 1·3. 3. 50·5° and 89·5. 4. £140, £160.
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